Synchronization Condition of Power Networks by Using Non-Uniform Kuramoto Model

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Abstract—This paper discusses the synchronization problem and it's solutions for large scale power network systems. We analyze dynamical properties of power networks and derive the synchronization condition of electrical generators. First, we employ a termed Kron reduction method to derive swing equation, which describes states of generators. Second, the derived equation is reduced into Non-uniform Kuramoto model, and then derive a synchronization condition by using the reduced model. Finally, we have succeed in relaxing the synchronization condition by controlling power units. Effectiveness of the proposed condition is shown via numerical control simulations.

1. Introduction

The power network system is one of the most complex and largest machine engineered by humankind in the 20th century. If troubles occur in this huge, complex system, it may result in large power failure and the loss is inscrutable. Therefore, the power grid is a system that should value safety most.

One form of power network stability is the transient stability, which is the ability of a power system to remain in synchronized state when subject to large transient disturbances such as loss of load, generator, or system components.

In order to analyze this transient stability various sophisticated methods have been proposed, for example [1], [2]. Unfortunately, the existing methods can cope only with simplified models and do not result in simple conditions to check if a power system synchronizes for a given network state and parameters. In fact, it is an outstanding problem to relate synchronization and transient stability of a power network to the underlying network parameters, state, and topology.

From such a background a quite new method of evaluating the transient stability of the power system has been proposed [3]. The feature of this method is to treat mathematical model of power network as approximation model called non-uniform Kuramoto model. This approximation model partly has the same character as the coupled Kuramoto oscillators, which is widely studied by the physics and the dynamical systems communities. In a word, non-uniform Kuramoto model plays a key role of the mediation of power network and coupled Kuramoto oscillators. Because synchronization condition of Kuramoto oscillators is well-known, this condition can be applied to power network via non-uniform Kuramoto model. The most different and remarkable point of this method is that the final synchronization condition is concise and the condition relate to the underlying network parameters, state, and topology.

In this paper, we employ this approximation method to model power grids. The reason why we did not employ the conventional method is that we consider that conciseness of evaluation method becomes important because power grid become more and more complex in the future. The contribution of this paper is to relax the synchronization condition derived from non-uniform Kuramoto model by controlling the generators connected in the power network.

First, we derive mathematical model of power network. The generators of the system are all controlled, to be precise, the controlled parameters are mechanical input of each generator. Second, a non-uniform Kuramoto model is derived from the power network model. Then, we derive and propose the synchronization condition and compare proposed condition and traditional one, then confirm numerical relaxing the synchronization condition. Finally, effectiveness of the proposed condition is shown via simulation results.

2. Power Network Model and Non-Uniform Kuramoto Model

2.1. The Mathematical Model of a Power Network

In a power network with N generators, we associate with generator i its normalized inertia constant H_i, its electrical power output $P_{e,i}$, its mechanical power input $P_{m,i}$, its damping constant D_i, and its rotor angle δ_i measured with respect to a synchronously rotating reference frame with frequency f. These parameters are given in each power network and the synchronous frequency f is typically given as 50Hz or 60Hz. Then, swing equations which are the rotor dynamics of generator i are given as

$$\frac{H_i}{\pi f}\ddot{\delta} = P_{m,i} - D_i\dot{\delta} - P_{e,i}, \quad i \in \{1, ..., N\} \tag{1}$$

Where all terms are in per unit value. Although a huge number of generator is generally connected to power network. In this paper, a power network which is consist of 8(= N) generators and 22 buses is employed as a controlled plant and its schematic structure is shown in Figure 1 [4]. This figure is partly based on New England Power Grid.

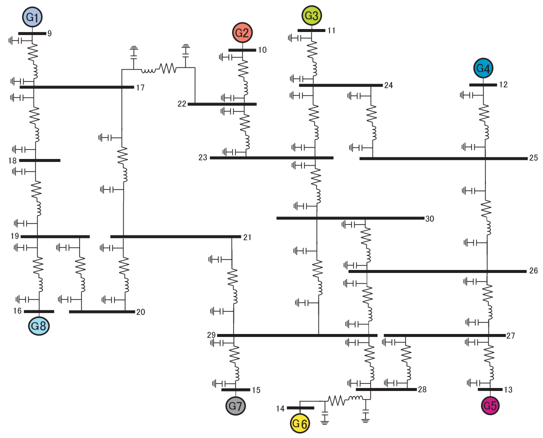


Fig 1: Power Grid

Defining Y as the admittance matrix of this power grid, this power network's each edge connecting two nodes is weighted by a non-zero admittance $Y_{ij} \in \mathbb{C}$. Each bus i is connected to the ground via a shunt admittance $Y_{ground-i}$. If a $Y_{ground-i}$ is zero, the bus i is said to be floating. Then we assume that all buses are floating. For this assumption, from a view point of circuit theory all buses of power network can be eliminated and Y results in the reduced admittance matrix Y_{red}. This reduction process is said to be Kron reduction or simply consecutive elimination of floating nodes [3].

Given N generators and M buses, the admittance matrix $Y' \subset \mathbb{C}^{(N+M-1)\times(N+M-1)}$ resulted from eliminating the kth bus from the admittance matrix $Y \subset \mathbb{C}^{(N+M)\times(N+M)}$ is given as

$$Y'_{ij} = Y_{ij} - \frac{Y_{ik}Y_{jk}}{Y_{kk}}, \quad i, j \in \{1, ..., N-1\} \tag{2}$$

The reduced admittance matrix Y_{red} is resulted from elimination of all buses, and relationship between each power unit is obvious by this matrix. Then, using the fact that the element $Y_{red\{ij\}}$ is equal to $G_{ij} + jB_{ij}$, electrical power output $P_{e,i}$ is

$$P_{e,i} = -\sum_{j\neq i}^{N} E_iE_j \left\{G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)\right\} \tag{3}$$

where G_{ij} and B_{ij} are the conductance and susceptance between generator i and j, and E_i is internal voltage of generator i. Then, we assume that this power network is lossless model for ultrahigh-voltage power transmission, so the mathematical model of a power network is derived from model (1), power output (3), and these assumptions.

$$\frac{H_i}{\pi f}\ddot{\delta} = P_{m,i} - K_{P,i}\dot{\delta} - D_i\dot{\delta} - E_i^2 G_{ii} - \sum_{j\neq i}^{N} E_iE_jB_{ij}\sin(\delta_i - \delta_j) \tag{4}$$

Where $K_{P,i}$ is feedback gain, and $P_{m,i} - K_{P,i}\dot{\delta}$ is the mechanical power input via feedback control. In this paper this model (4) is called the power network model.

2.2. The Approximation Model of the Power Network Model

This section puts power network model on mathematical ground, then derives approximation model of that model. The approximation model is named non-uniform Kuramoto model. The most discriminative properties of the model are that the model behaves not only as the power network model but also as the Kuramoto model. So this approximation model serves as a bridge between power network model and Kuramoto model. This leads to enable to analyze power network model as a oscillating system in terms of the Kuramoto model and apply the synchronization condition of the Kuramoto model to the power network model.

In order to derive the approximation model, we define the natural frequency $\omega_i := P_{m,i} - E_iG_{ii}^2$ (effective power input to generator i), the coupling weights $P_{ij} := E_iE_j \mid Y_{red\{ij\}} \mid$ (maximum power transferred between generators i and j), the singular perturbation parameter $\varepsilon := H_{max}/\pi f D_{min}$ (the worst-case choice of $H_i/\pi f D_i$), $F := (D_i/D_{min})/(M_i/M_{max})$ which will determine the speed of convergence of power network system (4). Then, with ω_i, P_{ij}, ε, and F_i the power network system (4) can be rewritten as

$$\dot{\delta}_i := \Omega_i \tag{5}$$

$$\varepsilon\dot{\Omega}_i = -F_i\Omega_i + \frac{F_i}{D_i + K_{P,i}}(\omega_i - \sum_{j\neq i}^{N} P_{ij}sin(\delta_i - \delta_j)) \tag{6}$$

Note that the gain $K_{P,i}$ which is mechanical power input in physical interpretation is involved in damping constant in this mathematical interpretation. In the system (6) it is possible to approximate $\varepsilon\dot{\Omega}_i$ by zero because the system (6) converges quickly as compared with system (5) and ε is sufficiently small. ε takes the value of about 0.01 in 50[Hz] or 60[Hz] areas. Then systems (5)-(6) can be transformed into

$$F_i\dot{\delta}_i = \frac{F_i}{D_i + K_{P,i}}\left\{\omega_i - \sum_{j\neq i}^{N} P_{ij}sin(\delta_i - \delta_j)\right\} \tag{7}$$

Following these procedures, the approximation model is finally derived as

$$(D_i + K_{P,i})\dot{\delta}_i = \omega_i - \sum_{j\neq i}^{N} P_{ij}\sin(\delta_i - \delta_j) \tag{8}$$

This model (8) is called non-uniform Kuramoto model.

3. Synchronization Condition

This section states our main result on the power network model and the non-uniform Kuramoto model.

Theorem 1 *Consider the power network model* (4) *and the non-uniform Kuramoto model* (8)*. Assume that models meet the synchronization condition (9) and initial condition (10).*

$$\frac{2E^2}{R} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i'} - \frac{\omega_j}{D_j'} \right\} \tag{9}$$

$$\theta(0) \in [0, \pi/2 - \gamma) \tag{10}$$

Where D_i' is $(D_i + K_{P,i})/(D_{\max} + K_{\max})$, R is the effective resistance[5] and γ is $(2E^2/R)/\left\{\max_{\{i,j\}}(\omega_i/D_i' - \omega_j/D_j')\right\}$. Then following (A)-(C) are ensured.

For the non-uniform Kuramoto model,
(A) Phase Locking: $\max_{\{i,j\}} \mid \theta_i - \theta_j \mid \leq \pi/2 - \gamma$
(B) Frequency Entrainment: $\dot{\theta}_{min}(0) \leq \dot{\theta}_{\infty} \leq \dot{\theta}_{max}(0)$
For the power network model,
(C) Approximation Error: $\theta_P(t) - \theta_N(t) = O(\varepsilon), \quad \forall t \geq 0$
where $\theta_P(t)$ and $\theta_N(t)$ are the phase of power network model and non-uniform Kuramoto model respectively.

Proof. The synchronization condition is derived from the theory of Kuramoto oscillator. The effective resistance or resistance distance R_{ij} between two nodes i, j of an undirected, connected, and uniformly weighted graph with Laplacian L is defined as

$$R_{ij} := (e_i - e_j)^T L^{\dagger}(e_i - e_j) = L_{ii}^{\dagger} + L_{jj}^{\dagger} - 2L_{ij}^{\dagger} \tag{11}$$

where e_i is defined as the vector of zeros of appropriate dimension with entry 1 at position i and e_j is in the same way. And $L^{\dagger}$ is the Moore-Penrose pseudo inverse of L. Since $L^{\dagger}$ is symmetric and $R_{ii} := 0$ by definition, the resistance matrix R is again a symmetric matrix. The effective resistance captures global properties of the graph topology such as distance and connectivity measures. Many interesting results relating R, L and $L^{\dagger}$ are found in previous work. Particularly, there is a notable result that if the number of nodes are ∞ their effective resistances R_{ij} converge to the constant effective resistance R which is used in synchronization condition (9). The following examples demonstrate that uniform resistances among a set of nodes occurs for various graph topologies, where we assumed uniform weightings for simplicity. First, in trivial case, if the nodes are 1-connected leaves of a highly symmetric graph among the buses, such as a star-shaped tree, a complete graph, or a combination of these two, then the effective resistance among the generator nodes is uniform. Second, the effective resistance in large-scale small-world networks is known to become uniform among sufficiently distant nodes [6]. Third and finally, geometric graphs such as lattices and their fuzzes are special random geometric graphs with vertices sampled on a grid. According to the previous arguments, the resistance among sufficiently distant lattice nodes becomes uniform in the large limit. □

This Theorem 1 states that if two models meet condition (9) and (10), non-uniform Kuramoto model synchronizes, and power network model can be approximated by non-uniform Kuramoto model with imperceptible error, that is, although approximation error which is of order ε occurs, power network synchronizes.

In order to compare this proposed synchronization condition with traditional one, we rewrite condition (9).

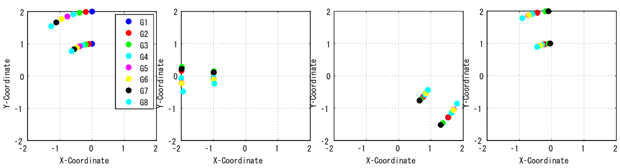
$$\frac{2E^2}{RD_{max}} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i' D_{max}} - \frac{\omega_j}{D_j' D_{max}} \right\} \tag{12}$$

$$\frac{2E^2}{RD_{max}} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right\} \tag{13}$$

where condition (12) is the rewritten condition and (13) is the traditional one [3]. The difference between two conditions is clearly the damping constant $D \leftrightarrow D' D_{max}$. It is also clear that the proposed condition is more relaxed than the traditional one because $D' D_{max}$ contains the gain K_P which is variable. Giving a kind of interpretation of this relaxation from the point of view of sets, since the fact that $D \subset D' D_{max}$ is clear, it is also clear that traditional condition (13) is included in the proposed one (12).

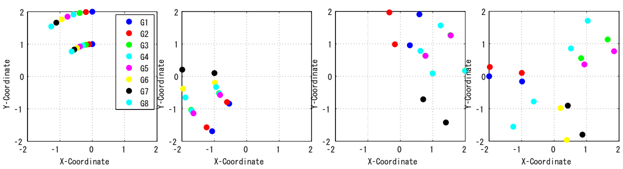
4. Simulation Results

This section deals the main results of the proposal condition via simulation results. Figure 2-(a) shows the simulation results that the power network is in conditions (9)-(10). On the other hand, Figure 2-(b) shows ones when the conditions (9)-(10) are not achieved.



(a1)t=0[s] (a2)t=1.0[s] (a3)t=3.0[s] (a4)t=10.0[s]

(a) In Condition



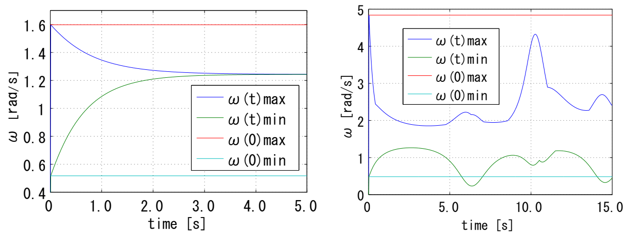
(b1)t=0[s] (b2)t=1.0[s] (b3)t=3.0[s] (b4)t=10.0[s]

(b) Out of Condition

Fig 2: Phase Shift

where these figures show the phase shift of the generators connecting to the power network such as Figure 1. In these figures, the inside dot sequence is a point to have projected the same point of the generators, for example N pole, on the X-Y coordinate with center axis of all dynamos Z axis and the outside one is the approximation phases of the inside same color phases. Hence the inside dot sequence behaves as a power network model (4) and the outside one behaves as a non-uniform Kuramoto model (8).

The validity concerning phase locking (A) of the Theorem 1 can visually be confirmed in this Figure 2. In this simulation, the result of frequency entrainment, (B) of the Theorem 1, can be seen in Figure 3.

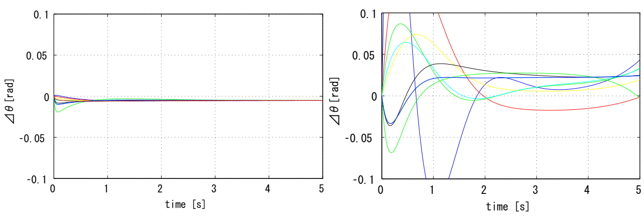


(a) In Condition (b) Out of Condition

Fig 3: Frequency Entrainment

From the Figure 3-(a), it can be confirmed that the minimum angular frequency and the maximum one converge between the minimum initial angular frequency and the maximum one. That is, it can be confirmed that angular frequency meets $\dot{\theta}_{min}(0) \leq \dot{\theta}_{\infty} \leq \dot{\theta}_{max}(0)$ in case that the power network is in conditions (9)-(10). This is frequency entrainment that is (B) of the Theorem 1. On the other hand, in Figure 3-(b), it can be seen that angular frequency does not meet frequency entrainment because the angular frequency does not converge and the minimum angular frequency periodically falls below the minimum initial one.

The following figure shows approximation error of phases between power network model (4) and non-uniform Kuramoto model (8) in case that the power network is in conditions and out of condition.



(a) In Condition (b) Out of Condition

Fig 4: Approximation Error

where $\Delta\theta$ is $\theta_P(t) - \theta_N(t)$. This is the phase approximation error between the power network model and the non-uniform Kuramoto model.

In this paper, the phase approximation error is evaluated based on whether $\theta_P(t) - \theta_N(t)$ is smaller than $O(\varepsilon)$ for all t. From the system parameters and the fact that the frequency is 50[Hz] or 60[Hz] in Japan, USA, and so on, ε is an order of 10^{-1}. Therefore, the approximation error is evaluated based on whether $\theta_P(t) - \theta_N(t)$ is smaller than 10^{-1} for all t. As evaluating approximation like this, from the Figure 4-(a), it can be confirmed that if the power network is in conditions the approximation is accurate. On the other hand, if the power network is out of conditions an opposite phenomena appears in the Figure 4-(b).

5. Conclusions

This paper studied synchronization in a network-preserving power system model from the background of the technical development of smart grid, the complication progress of the electric power system and the increasing dependence on renewable energy, such as wind and solar power.

We analyzed the dynamical properties of the power networks by using Kron reduction, effective resistance of the power grid and non-uniform Kuramoto model theory. Consequently, the synchronization condition (9) and the initial condition (10) were derived, where the mechanical inputs of the generators was controlled as $P_{m,i} - K_{P,i}\dot{\delta}$. Then, we discussed the differences between the proposed condition and previous one and confirmed the numerical relaxing of the synchronization condition. Finally, we showed the validity of the proposed conditions via simulation results. The simulation results of the phase shift, frequency entrainment and approximation error were seen in the Fig.2-4 respectively.

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