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An $\mathcal{H}_\infty$ Control Considering Initial State Uncertainties of Controllers

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***Abstract*— This paper deals with an $\mathcal{H}_\infty$ control attenuating initial-state uncertainties of controllers. An $\mathcal{H}_\infty$ control problem, which treats a mixed attenuation of disturbance and initial-state uncertainty of controllers for linear time-invariant systems in the infinite-horizon case, is examined. The mixed attenuation supplies $\mathcal{H}_\infty$ controls with good transients and assures $\mathcal{H}_\infty$ controls of robustness against initial-state uncertainty of controllers. We derive a necessary and sufficient condition of the mixed attenuation problem. Furthermore we apply this proposed method to a magnetic suspension system, and evaluate attenuation property of the proposed an $\mathcal{H}_\infty$ control approach.**

***Keywords*— $\mathcal{H}_\infty$ Control, Robust Control, Initial-State Uncertainties of Controllers, DIA**

I. Introduction

$\mathcal{H}_\infty$ control is one of the prevailing control system design methods which have been applied to various industries including automobile, space development and so on. It was established as a powerful robust control system design tool. A general $\mathcal{H}_\infty$ control for linear time-invariant systems attenuates the effect of disturbances on controlled outputs and is originally defined under the assumption that the initial states of the system are zero. In actual plants, the initial states are often uncertain and might be zero or non-zero. If the initial states are non-zero, the system adopting an $\mathcal{H}_\infty$ control will present some transients as the effect of the non-zero initial states, to which the $\mathcal{H}_\infty$ control is not intrinsically responsible. Recently, gain-scheduling controls and switching controls have been researched strenuously[1][2][3] for high functional control such as global stabilization and fault tolerant properties. It is well known that bumpy responses are caused after controller switching and it is a serious issue. Part of the reason for this problem are states of systems just behind switching act the uncertain initial states for the switched controller. These motivates us in this paper to be concerned with $\mathcal{H}_\infty$ controls which accomplish a mixed attenuation of disturbance and initial-state uncertainty of the controller.

It is expected that the mixed attenuation of disturbance and initial-state uncertainty in controlled outputs supplies $\mathcal{H}_\infty$ controls with some good transients and assures $\mathcal{H}_\infty$ controls of robustness against initial-state uncertainty. In the finite-horizon case, a generalized type of $\mathcal{H}_\infty$ control problem was formulated and solved[4][5], and was extended to the infinite-horizon case[5][6]. The problem discussed in [6] was limited to time-invariant systems satisfying the orthogonality assumptions[7][8][9]. In [10], an infinite-horizon disturbance and initial state uncertainty attenuation control problem without the orthogonality assumptions had formulated and a necessary and sufficient condition for a solution, together with an explicit formula of the solution, is derived. Moreover this approach was applied to a magnetic suspension system, and evaluated the effectiveness[10].

The previous mixed attenuation problem is formulated only for the initial-states of the plants. However, Initial-states uncertainties of controllers also might cause bumpy responses by the controller switching. In this paper, we have formulated an infinite-horizon disturbance and initial state uncertainties of both of plants and controllers attenuation control problem. A necessary and sufficient condition for a solution to exist is derived. Finally we apply this approach to a magnetic suspension system, and evaluate the effectivity against robustness for initial-state uncertainties by control simulation.

II. Preliminaries

Consider the linear time-invariant system which is defined on the time interval $[0,\infty)$ and described by

$$\left\{\begin{array}{rcl} \dot{x} &=& Ax + B_1 w + B_2 u, \quad x(0) = x_0 \\ z &=& C_1 x + D_{12} u \\ y &=& C_2 x + D_{21} w \end{array}\right. \tag{1}$$

where $x \in R^n$ is the state and $x(0) = x_0$ is the initial state; $u \in R^r$ is the control input; $y \in R^m$ is the observed output; $z \in R^q$ is the controlled output; $w(t) \in R^p$ is the disturbance and is a square integrable function define on $[0,\infty)$.

$A, B_1, B_2, C_1, C_2, D_{12}$ and D_{21} are constant matrices of appropriate dimensions and satisfy that

- (A, B_1): Stabilizable, (C_1, A): Detectable
- (A, B_2): Controllable, (C_2, A): Observable
- $D_{12}^T D_{12} = I, \quad D_{21} D_{21}^T = I$
- $D_{12}^T C_1 = 0, \quad B_1 D_{21}^T = 0$

For system (1), every admissible control $u(t)$ is given by a linear time-invariant system of the form

$$\left\{\begin{array}{rcl} u &=& J\underline{x} + Ky \\ \dot{\underline{x}} &=& G\underline{x} + Hy, \quad \underline{x}(0) = \underline{x}_0 \end{array}\right. \tag{2}$$

which makes the closed-loop system given by (1) and (2) internally stable, where $\underline{x}$ is the state of the controller of a finite dimensions; J, K, G and H are constant matrices of appropriate dimensions.

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III. An $\mathcal{H}_\infty$ control considering initial-state uncertainties of Plants

In this section, previous results; an $\mathcal{H}_\infty$ control considering initial-state uncertainties of Plants[10] is shown. Let (1), (2) be Plant and controller, where there initial-states are $x(0) = x_0,\ \underline{x}(0) = 0$ respectively. Note that there is a no consideration about initial-states of controllers.

For the system and the class of admissible controls described above, consider a mixed-attenuation problem.

Problem 1. $\mathcal{H}_\infty$DIA Control Problem

Find an admissible control attenuating disturbances and initial state uncertainties in the way that, for given $N_1 > 0$,

$$\|z\|_2^2 < \|w\|_2^2 + x_0^T N_1^{-1} x_0 \tag{3}$$

for all $w \in L^2[0,\infty)$ and all $x_0 \in R^n$, s.t., $(w, x_0) \neq 0$. We call such an admissible control the **D**isturbance and **I**nitial state uncertainty **A**ttenuation (**DIA**) control. The weighting matrix N_1 on x_0 is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation. This problem is a kind of mixed attenuation problems[11], [12].

Let us assume the so-called Riccati-based conditions as below in order to solve the DIA control problem.

(A1) There exists a solution $M > 0$ to the Riccati equation

$$MA + A^T M - M(B_2B_2^T - B_1B_1^T)M + C_1^T C_1 = 0 \tag{4}$$

such that

$$A - B_2B_2^T M + B_1B_1^T M \tag{5}$$

is stable.

(A2) There exists a solution $P > 0$ to the Riccati equation

$$AP + PA^T - P(C_2^T C_2 - C_1^T C_1)P + B_1B_1^T = 0 \tag{6}$$

such that

$$A - PC_2^T C_2 + PC_1^T C_1 \tag{7}$$

is stable.

(A3) $S := M(I - PM)^{-1} > 0$

Remark 1. (A3) is equal to $\rho(PM) < 1$. Where $\rho(X)$ denotes the spectral radius of matrix X, and $\rho(X) = \max|\lambda_i(X)|$.

Theorem 1.[6], [10] Suppose that the conditions (A1),(A2) and (A3) are satisfied. The $\mathcal{H}_\infty$ central controller satisfied the condition (3) if and only if the condition (A4) is satisfied. Where the $\mathcal{H}_\infty$ central controller is as below.

$$\begin{cases} u = -B_2^T S\underline{x} \\ \dot{\underline{x}} = A\underline{x} + B_2 u + PC_2^T(y - C_2\underline{x}) + PC_1^T C_1 \underline{x} \end{cases} \tag{8}$$

(A4) $Q + N_1^{-1} - P^{-1} > 0$

where Q is the maximal solution of the Riccati equation

$$\begin{aligned} &Q(A + B_1B_1^T P^{-1}) + (A + B_1B_1^T P^{-1})^T Q \\ &-Q(B_1^T - D_{21}^T C_2 PL)^T(B_1^T - D_{21}^T C_2 PL)Q = 0 \end{aligned} \tag{9}$$

with $L := (I - PM)^{-1}$.

IV. An $\mathcal{H}_\infty$ control considering initial-state uncertainties of Controllers

An $\mathcal{H}_\infty$ control considering initial-state uncertainties of Controller; main result of this paper is shown in this section. Suppose initial states of Plants and Controllers are respectively $x(0) = 0,\ \underline{x}(0) = \underline{x}_0$.

Here consider a mixed attenuation problem as below.

Problem 2. $\mathcal{H}_\infty$ control problem considering initial-state uncertainties of Controllers

Find an admissible control attenuating disturbances and initial state uncertainties in the way that, for given $N_2 > 0$,

$$\|z\|_2^2 < \|w\|_2^2 + \underline{x}_0^T N_2^{-1} \underline{x}_0 \tag{10}$$

for all $w \in L^2[0,\infty)$ and all $\underline{x}_0 \in R^n$, s.t., $(w, x_0) \neq 0$.

The weighting matrix N_2 on $\underline{x}_0$ is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation. A larger choice of N_2 in the sense of matrix inequality order means finding an admissible control which attenuates the initial-state uncertainty more.

For this problem under the Riccati-based three conditions (A1),(A2) and (A3), we can obtain the following result.

Lemma 1. Suppose that the condition (A1),(A2) and (A3) are satisfied, then the $\mathcal{H}_\infty$ central controller (8) satisfies the following inequality

$$\|z\|_2^2 < \|w\|_2^2 + \underline{x}_0^T (S + P^{-1}) \underline{x}_0 \tag{11}$$

for all $w \in L^2[0,\infty)$ and all $\underline{x}_0 \in R^n$.

Proof.

Consider the functional $V_1(t)$,

$$V_1(t) := \underline{x}^T S \underline{x} + (x - \underline{x})^T P^{-1} (x - \underline{x}) \tag{12}$$

then, differentiating (12) both sides with respect to t, and inserting conditions (A1)-(A3) into the right hand side, we have

$$\begin{aligned} \dot{V}_1(t) = &-\|z\|^2 + \|w\|^2 + \|u + B_2^T S\underline{x}\|^2 \\ &- \|w - w_0\|^2 \end{aligned} \tag{13}$$

where $w_0 := D_{21}^T C_2 PS\underline{x} + (B_1^T - D_{21}^T C_2 P)P^{-1}(x - \underline{x})$.

Integrating both sides with respect to t over the interval $[0, \infty)$, we obtain the left hand side as

$$\int_0^\infty \dot{V}_1(t)dt = -\underline{x}_0^T (S + P^{-1}) \underline{x}_0 \tag{14}$$

implying the control input $u(t)$ as (8), and

$$-\underline{x}_0^T(S+P^{-1})\underline{x}_0 = -\|z\|_2^2 + \|w\|_2^2 - \|w-w_0\|_2^2 \quad (15)$$

from $\|w-w_0\|_2^2 > 0$, we finally obtain as

$$\|z\|_2^2 < \|w\|_2^2 + \underline{x}_0^T(S+P^{-1})\underline{x}_0$$

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This Lemma is concerned with the condition for P and S, not N_2. Next, the following condition is assumed.

(A5) $S + P^{-1} < N_2^{-1}$

If the condition (A5) holds, the inequality (10) follows from the inequality (11).

$$\begin{aligned}\|z\|_2^2 &< \|w\|_2^2 + \underline{x}_0^T(S+P^{-1})\underline{x}_0 \\ &< \|w\|_2^2 + \underline{x}_0^T N_2^{-1}\underline{x}_0 \end{aligned} \quad (16)$$

In view of the inequality above, the condition (A5) seems necessary for the $\mathcal{H}_\infty$ central controller satisfied (10).

Next we will show a necessary and sufficient condition which the main result on this paper. In order to state the result, let us introduce the following condition:

(A6) $L^TQL + N_2^{-1} - P^{-1} - S > 0$

where Q is the maximal solution of Riccati equation (9).

Theorem 2. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The $\mathcal{H}_\infty$ central controller satisfied the condition (10) if and only if the condition (A6) is satisfied.

Proof. We prove Lemma2 and Lemma3. Then Theorem2 follows. Lemma2 and Lemma3 require the following condition:

(A7) For all $w \in L^2[0,\infty)$ and all $\underline{x}_0 \in R$ s.t., $(w, \underline{x}_0 \neq 0)$, the inequality

$$\|w-w_0\|_2^2 + \underline{x}_0^T(N_2^{-1} - P^{-1} - S)\underline{x}_0 > 0 \quad (17)$$

holds.

Lemma 2. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The $\mathcal{H}_\infty$ central controller (8) satisfied the condition (10) if and only if the condition (A7) is satisfied.

Proof. Consider the functional $V_1 = \underline{x}^T S \underline{x} + (x-\underline{x})^T P^{-1}(x-\underline{x})$, then, differentiating both sides with respect to t, and inserting (A1)-(A3) into the right hand side, and integrating both sides with respect to t over the interval $[0,\infty)$, we obtain

$$\|w-w_0\|_2^2 = \|w\|_2^2 - \|z\|_2^2 + \underline{x}_0^T(S+P^{-1})\underline{x}_0 \quad (18)$$

Insert (18) into (A7), then we have

$$\|z\|_2^2 < \|w\|_2^2 + \underline{x}_0^T N_2^{-1}\underline{x}_0 \quad (19)$$

Converse, insert (18) into (10), then we have

$$\|w-w_0\|_2^2 + \underline{x}_0^T(N_2^{-1} - P^{-1} - S)\underline{x}_0 > 0 \quad (20)$$

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Lemma 3. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The condition (A7) is equivalent to the condition (A6).

Proof. Consider the functional $V_2(t) := f^TQf$, where $f := x(t) - L\underline{x}(t)$. Differentiating both sides with respect to t and completing the square argument as

$$\dot{V}_2(t) = \|(w-w_0) + (B_1^T - D_{21}^T C_2 P L^T)Qf\|^2 - \|w-w_0\|^2 \quad (21)$$

then, integrating both sides with respect to t over the interval $[0,\infty)$, we have left hand side as

$$\int_0^\infty \dot{V}_2(t)dt = -\underline{x}_0^T L^T Q L \underline{x}_0 \quad (22)$$

implying the control input $u(t)$ as (8), and

$$\begin{aligned}-\underline{x}_0^T L^T Q L \underline{x}_0 = \|w-w_0 + (B_1^T - D_{21}^T C_2 P L^T)Qf\|_2^2 \\ - \|w-w_0\|_2^2 \end{aligned} \quad (23)$$

Insert (23) into (A7), then we have

$$\begin{aligned}\|w-w_0 + (B_1^T - D_{21}^T C_2 P L^T)Qf\|_2^2 \\ + \underline{x}_0^T(L^TQL + N_2^{-1} - P^{-1} - S)\underline{x}_0 > 0 \end{aligned} \quad (24)$$

The 1st term in the left hand side are positive, hence we have

$$L^TQL + N_2^{-1} - P^{-1} - S > 0 \quad (25)$$

Converse, insert (23) into (A6), then we have

$$\|w-w_0\|_2^2 + \underline{x}_0^T(N_2^{-1} - P^{-1} - S)\underline{x}_0 > 0 \quad (26)$$

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V. An $\mathcal{H}_\infty$ Control Considering Initial-State Uncertainties of Plants and Controllers

In this section, we formulated an $\mathcal{H}_\infty$ control problem considering initial states of both plants and controllers, and a necessary and sufficient condition for a solution to exist is showed. Suppose initial states of Plants and Controllers are $x(0) = x_0,\ \underline{x}(0) = \underline{x}_0$ respectively . In order to handle initial states synthetically, we introduce the state of closed-system $\hat{x}(t) := \begin{bmatrix} x(t)^T & \underline{x}(t)^T \end{bmatrix}^T$; (initial state of closed-system $\hat{x}(0) := \hat{x}_0 = \begin{bmatrix} x_0^T & \underline{x}_0^T \end{bmatrix}^T$).

Here consider a mixed attenuation problem sated as below.

Problem 3. $\mathcal{H}_\infty$ control problem considering initial-state uncertainties of Plants and Controllers

Find an admissible control attenuating disturbances and initial state uncertainties in the way that, for given $N_3 > 0$,

$$\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T N_3^{-1}\hat{x}_0 \quad (27)$$

for all $w \in L^2[0,\infty)$ and all $\hat{x}_0 \in R^{2n}$, s.t., $(w, \hat{x}_0) \neq 0$.

The weighting matrix N_3 on $\hat{x}_0$ is a measure of relative importance of the initial-state uncertainty of closed-loop system attenuation to the disturbance attenuation. A larger choice of N_3 means finding an admissible control which attenuates the initial-state uncertainty of closed-loop system more.

For this problem, we can obtain the following result.

Lemma 4. Suppose that the condition (A1),(A2) and (A3) are satisfied, then the $\mathcal{H}_\infty$ central controller (8) satisfies the following inequality

$$\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T M_{PS} \hat{x}_0 \qquad (28)$$

for all $w \in L^2[0,\infty)$ and all $\hat{x}_0 \in R^{2n}$.
where, $M_{PS} := \begin{bmatrix} P^{-1} & -P^{-1} \\ -P^{-1} & S+P^{-1} \end{bmatrix} \in R^{2n\times 2n}$.

Proof. Consider the functional $V_1(t) = \underline{x}^T S \underline{x} + (x - \underline{x})^T P^{-1}(x-\underline{x})$ again. Then, differentiating both sides with respect to t, and inserting conditions (A1)-(A3) into the right hand side in the same way of previous section, then we have

$$\dot{V}_1(t) = -\|z\|^2 + \|w\|^2 + \|u + B_2^T S\underline{x}\|^2 - \|w - w_0\|^2$$

Integrating both sides with respect to t over the interval $[0,\infty)$, we have left hand side as

$$\begin{aligned}\int_0^\infty \dot{V}_1(t)dt &= -[x_0^T \ \underline{x}_0^T]^T \begin{bmatrix} P^{-1} & -P^{-1} \\ -P^{-1} & S+P^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \underline{x}_0 \end{bmatrix} \\ &= -\hat{x}_0^T M_{PS} \hat{x}_0 \end{aligned} \qquad (29)$$

implying the control input $u(t)$ as (8),

$$-\hat{x}_0^T M_{PS} \hat{x}_0 = -\|z\|_2^2 + \|w\|_2^2 - \|w - w_0\|_2^2 \qquad (30)$$

here from $\|w - w_0\|_2^2 > 0$, we obtain as

$$\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T M_{PS} \hat{x}_0$$

■

Lemma 4 is concerned with the condition for matrix M_{PS}, not for N_3. Next the following condition is assumed.
(A8) $M_{PS} < N_3^{-1}$
If the condition (A8) holds, the inequality (27) follows from the inequality (28).

$$\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T M_{PS} \hat{x}_0 < \|w\|_2^2 + \hat{x}_0^T N_3^{-1} \hat{x}_0 \qquad (31)$$

In view of the inequality above, the condition (A8) is necessary for the an $\mathcal{H}_\infty$ central controller satisfied the condition (27). Next we will show a necessary and sufficient condition. In order to state the result, let us introduce the following condition:
(A9) $M_{QL} + N_3^{-1} - M_{PS} > 0$
where $M_{QL} := \begin{bmatrix} Q & -QL \\ -L^T Q & L^T QL \end{bmatrix} \in R^{2n\times 2n}$.

Theorem 3. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The $\mathcal{H}_\infty$ central controller satisfied the condition (27) if and only if the condition (A9) is satisfied.

Proof. Theorem 3 can be proven in same way of Theorem 4 in the section 4. First, we prove Lemma4 and Lemma5. Then Theorem3 follows. Lemma4 and Lemma5 require the following condition:

(A10) For all $w \in L^2[0,\infty)$ and all $\hat{x}_0 \in R$ s.t., $(w, \hat{x}_0 \neq 0)$, the inequality below holds.

$$\|w - w_0\|_2^2 + \hat{x}_0^T (N_3^{-1} - M_{PS}^{-1}) \hat{x}_0 > 0 \qquad (32)$$

Lemma 5. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The an $\mathcal{H}_\infty$ central controller (8) satisfied the condition (27) if and only if the condition (A10) is satisfied.

Proof. Consider the functional $V_1 = \underline{x}^T S \underline{x} + (x - \underline{x})^T P^{-1}(x - \underline{x})$, then, differentiating both sides with respect to t, and inserting (A1)-(A3) into the right hand side, and integrating both sides with respect to t over the interval $[0,\infty)$, we obtain

$$\|w - w_0\|_2^2 = \|w\|_2^2 - \|z\|_2^2 + \hat{x}_0^T M_{PS} \hat{x}_0 \qquad (33)$$

Insert (30) into (A10), we have

$$\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T N_3^{-1} \hat{x}_0$$

Converse, insert (27) into (30), we have

$$\|w - w_0\|_2^2 + \hat{x}_0^T (N_3^{-1} - M_{PS}) \hat{x}_0 > 0$$

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Lemma 6. Suppose that the conditions (A1),(A2) and (A3) are satisfied. The $\mathcal{H}_\infty$ central controller (8) satisfied the condition (27) if and only if the condition (A10) is satisfied.

Proof. Consider the functional $V_2(t) = f^T Q f$ again. Then, differentiating both sides with respect to t, and inserting (A1)-(A3) into the right hand side, we obtain

$$\dot{V}_2(t) = \|(w - w_0) + (B_1^T - D_{21}^T C_2 P L^T) Q f\|^2 - \|w - w_0\|^2 \qquad (34)$$

Integrating both sides with respect to t over the interval $[0,\infty)$, we have left hand side as

$$\begin{aligned}\int_0^\infty \dot{V}_2(t)dt &= V_2(\infty) - V_2(0) \\ &= -\left[x_0^T \ \underline{x}_0^T\right] \begin{bmatrix} Q & -QL \\ -L^T Q & L^T QL \end{bmatrix} \begin{bmatrix} x_0 \\ \underline{x}_0 \end{bmatrix} \\ &= -\hat{x}_0^T M_{QL} \hat{x}_0 \end{aligned} \qquad (35)$$

Join (34) and (35),we have

$$\begin{aligned}-\hat{x}_0^T L^T QL \hat{x}_0 &= \|w - w_0 + (B_1^T - D_{21}^T C_2 P L^T) Q f\|_2^2 \\ &\quad - \|w - w_0\|_2^2 \end{aligned} \qquad (36)$$

Insert (36) to (A10), we have

$$\begin{aligned}&\|w - w_0 + (B_1^T - D_{21}^T C_2 P L^T) Q f\|_2^2 \\ &+ \hat{x}_0^T (M_{QL} + N_3^{-1} - M_{PS}) \hat{x}_0 > 0 \end{aligned} \qquad (37)$$

where 1st term of (37) are positive, hence we have

$$M_{QL} + N_3^{-1} - M_{PS} > 0$$

Converse, insert (36) in (A9), we have

$$\|w - w_0\|_2^2 + \hat{x}_0^T (N_3^{-1} - M_{PS}) \hat{x}_0 > 0$$

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VI. Summary and Discussion of the Results

The obtained three problems and their conditions are summarized in Table I.

The condition (A9) includes the conditions (A4) and (A6) in itself and this result looks reasonable. The interconnection of the plant and the controller is considered in the condition (A9) and it is a harder condition to achieve than (A4) and (A6).

TABLE I
Summary of Conditions

Initial value	Control Problem & Condition
x_0	$\\|z\\|_2^2 < \\|w\\|_2^2 + x_0^T N_1^{-1} x_0$ **(A4)** $Q + N_1^{-1} - P^{-1} > 0$
$\underline{x}_0$	$\\|z\\|_2^2 < \\|w\\|_2^2 + \underline{x}_0^T N_2^{-1} \underline{x}_0$ **(A6)** $L^T QL + N_2^{-1} - P^{-1} - S > 0$
$\hat{x}_0 = \begin{bmatrix} x_0 \\ \underline{x}_0 \end{bmatrix}$	$\\|z\\|_2^2 < \\|w\\|_2^2 + \hat{x}_0^T N_3^{-1} \hat{x}_0$ **(A9)** $M_{QL} + N_3^{-1} - M_{PS} > 0$

VII. Examples

A. Control System Design

The proposed method is applied to Magnetic Suspension Systems (MSS) and verified their effectivity by simulation. A mathematical model of MSS is described as below [10]

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g u_g + D_g v_0 \\ y_g &= C_g x_g + w_0 \end{aligned} \tag{38}$$

where $x_g(t) := [x(t)\ \dot{x}(t)\ i(t)]^T,\ u_g(t) := e(t),\ v_0(t) := [v_m(t)\ v_L(t)]^T$, $x(t)$ is displacement of ball, $i(t)$ is current, $u_g(t) = e(t)$ is control input as voltage, $v_m(t), v_L(t)$ is disturbance and noises, $w_0(t)$ is sensor noises or influences of uncertainties. (A_g, B_g) and (A_g, D_g) is controllable and observable respectively.

Let us consider the disturbances v_0 and w_0. Since v_0 mainly acts on the plant in a low frequency, and w_0 shows an uncertainty caused via unmodeled dynamics. We introduced weighting functions $W_v(s)$ and W_w. $W_v(s)$ is a weighting function for v_0 with 1st order system or $W_v(s) = \frac{\mathrm{gain}\times\omega_n}{s+\omega_n}$, W_w is a weighting scalar for w_0.

We selected $x(t)$ and $\dot{x}(t)$ as a part of controlled output z_1, and selected $u_g(t)$ as an another part z_2. Then we introduced weighting matrix $\Theta = \mathrm{diag}\,[\theta_1\ \theta_2]$ on the regulated variables z_1, and weighting scalar ρ on the regulation of the control input $u(:= u_g)$.

Finally, let $x := \left[x_g^T\ x_w^T\right]^T$, where x_w denotes the state of $W_v(s)$, and $z := \left[z_1^T\ z_2^T\right]^T$, then we can construct the generalized plant as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{39}$$

$$\begin{aligned} A &= \begin{bmatrix} A_g & D_g C_w \\ 0 & A_w \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & D_g D_w \\ 0 & B_w \end{bmatrix}, \\ B_2 &= \begin{bmatrix} B_g \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} \Theta F_g & 0 \\ 0 & 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ \rho \end{bmatrix}, \\ C_2 &= [C_g \quad 0],\ D_{21} = [W_w \quad 0] \end{aligned}$$

Now our control problem setup is finding an admissible controller $K(s)$ that attenuates disturbances and initial state uncertainties to achieve DIA condition in (27). After some control design iterations, the design parameters; $W_v(s)$, W_w, Θ, and ρ are chosen appropriately, and a direct calculations yield the $\mathcal{H}_\infty$DIA controller $K(s)$.

In order to simplify numerical evaluations, a construction of the maximum value of the weighting matrix N_3 is limited and locked as below

$$N_3 = n_3 I \tag{40}$$

where n_3 is positive scalar and I is 8th order unit matrix. The inequality $\|z\|_2^2 < \|w\|_2^2 + \hat{x}_0^T N_3^{-1} \hat{x}_0$ in (27) suggests that there exists a trade-off between disturbance attenuation and initial state uncertainty attenuation, and weighting matrix N_3 is an index for a relative significance of initial state uncertainty attenuation against disturbance attenuation.

To verify effectiveness of N_3, we compare a controller designed based on N_3 (proposed method) to a controller designed based on N_1(previous method). Table II shows values of n_1, n_3 with changing of a design parameter ω_n in $W_v(s)$.

From Table II, let define K_{n1} as a controller with $n_1 = 2.60 \times 10^{-3}$($\omega_n = 2.65 \times 10^{-1}$), and K_{n3} as a controller with $n_3 = 2.60 \times 10^{-3}$($\omega_n = 1.0 \times 10^{-2}$). The frequency response of the controller K_{n3} and K_{n1} are shown in Fig.1 by a solid line and a dotted line respectively. Comparing these controllers, we can see that $K_{n3}(s)$ has a low gain at wide frequency range and high robustness.

TABLE II
n_1 and n_3 with changing ω_n

ω_n	n_1	n_3
1.0×10^{-3}	4.70×10^{-3}	2.35×10^{-3}
1.0×10^{-2}	5.20×10^{-3}	$\mathbf{2.60 \times 10^{-3}}$
1.0×10^{-1}	3.35×10^{-3}	1.68×10^{-3}
2.65×10^{-1}	$\mathbf{2.60 \times 10^{-3}}$	1.30×10^{-3}
1.0×10^{0}	1.77×10^{-3}	0.88×10^{-3}

B. Simulation and Evaluation I: N_1 and N_3

We have conducted 4 simulations (Disturbance responses and Initial responses with initial states of plant, controller and plant+controller) to evaluate properties of $K_{n3}(s)$.

An initial response for $x_g(0) := x_{g0} = [1.0{\times}10^{-3}\ 0\ 0]^T$ is shown in Fig.2. Responses of K_{n1} have short settling time, but have large overshoots and vibratory responses. In contrast, responses of K_{n3} have a small overshoot and converge smoothly. Initial responses for $\underline{x}_0 = [0\ 0\ 0\ 5.0 \times$

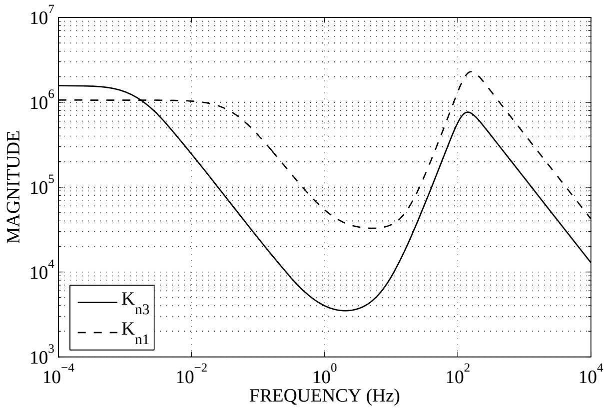

Fig. 1. Frequency Responses of $\mathcal{H}_\infty$DIA Controller

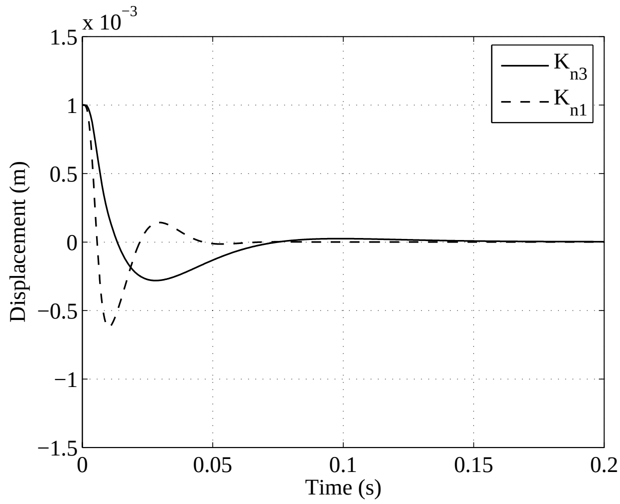

Fig. 2. Initial Responses (Plant)

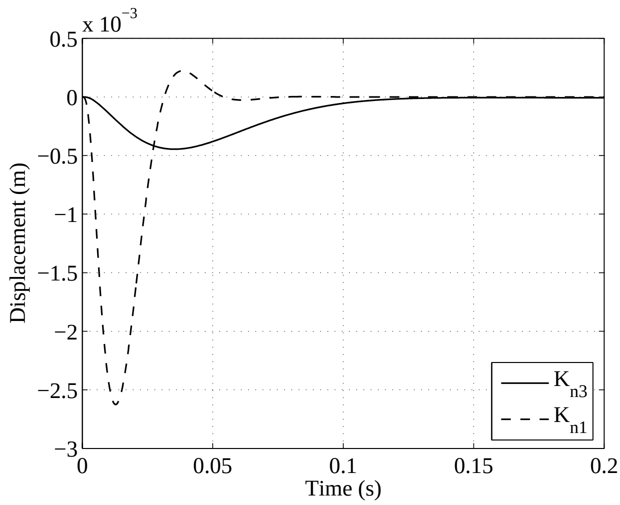

Fig. 3. Initial Responses (Controller)

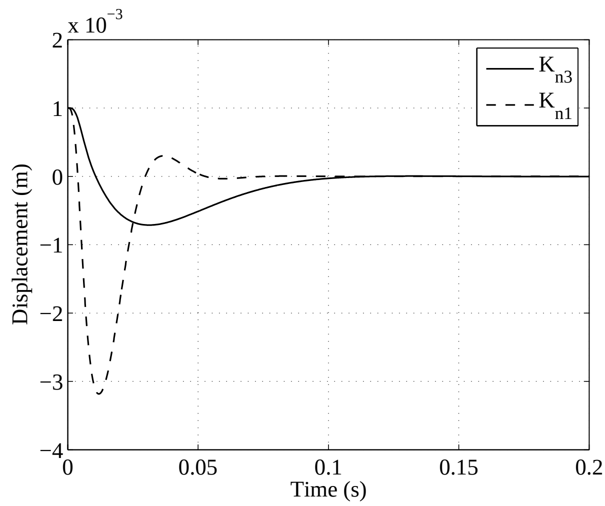

Fig. 4. Initial Responses (Plant+Controller)

$10^{-5}]^T$ and both x_{g0} and $\underline{x}_0$ are shown in Figs.3 and 4 respectively. These responses show similar characteristic to Fig.2. These results suggest K_{n3} has a good attenuation ability for initial state uncertainties than K_{n1}.

A disturbance response is shown in Fig.5, where input disturbance is a step-type force disturbance that has magnitude of about 25% steady-state attractive force(0.7[N]) directed downward. In this case, K_{n1} has a better transient performance than K_{n3}. The reason this result is that K_{n3} was selected to have a stronger attenuation performance for initial-state uncertainties giving up a performance of attenuation for disturbance.

C. Simulation and Evaluation II: Characteristic of N_3

Next we verify controller characteristic for a change of n_3. From Table II, K_{n3a}, K_{n3b} and K_{n3c} that has respectively $n_3 = 2.60\times10^{-3}$, 1.68×10^{-3} and 0.88×10^{-3} are defined. Frequency responses of each controller are shown in Fig.6. Characteristics of controller tend to have lower gain when controllers have higher n_3, especially frequency range is in $10^{-3} \sim 10^1$[Hz].

Initial responses and disturbance responses of K_{n3a}, K_{n3b} and K_{n3c} are shown in Figs.7 and 8. Where an initial value; $x_{g0} = [1.0\times10^{-3}\ 0\ 0]^T$, $\underline{x}_0 = [0\ 0\ 0\ 5.0\times10^{-3}]$ are given to systems. An initial responses in Fig.7, influences of initial state uncertainties are attenuated by K_{n3a}, K_{n3b} and K_{n3c} in that order, and that means a controller that has larger n_3 has a higher robustness for initial state uncertainties. On the other hand, disturbance responses in Fig.8, a response of K_{n3a} has the largest overshoot in these three examples. It can be seen that this result is appropriate at the view point of a condition inequality (27), because there exist a trade-off between initial state uncertainty attenuation and disturbance attenuation.

All these results considered, we reached the conclusion that the weighting matrix N_3 is an effective index for a relative significance of an initial state uncertainty attenuation against a disturbance attenuation.

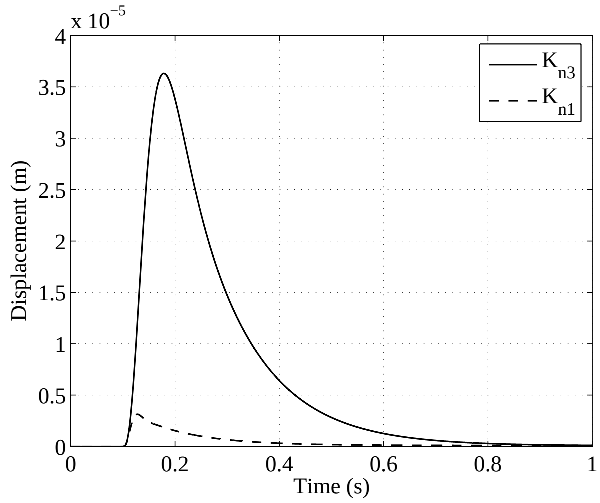

Fig. 5. Disturbance Responses

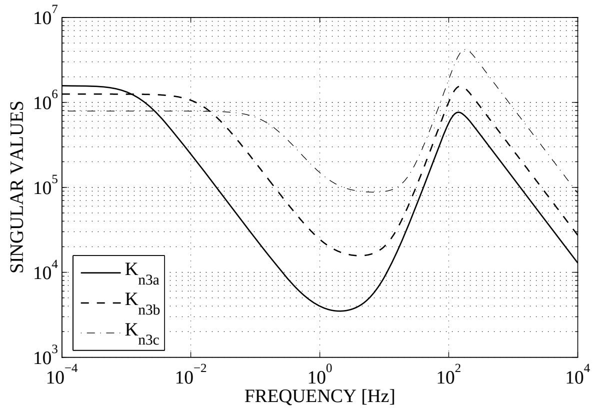

Fig. 6. Frequency Responses

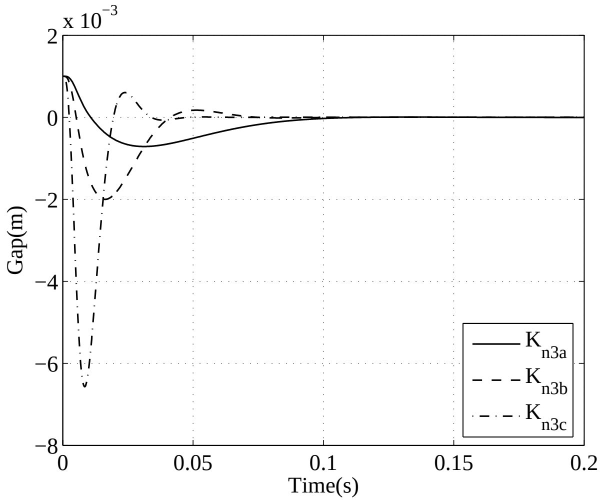

Fig. 7. Initial Responses

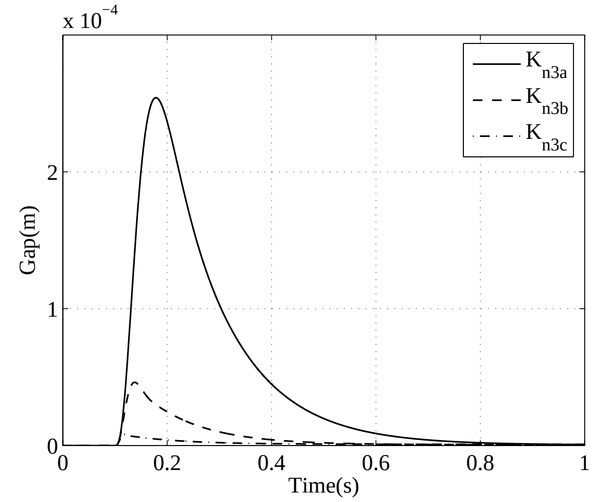

Fig. 8. Disturbance Responses

VIII. Conclusions

In this paper, we considered $\mathcal{H}_\infty$ control problems with attenuating initial state uncertainties of controllers.

First, we formulated an $\mathcal{H}_\infty$ control problem which considers a mixed attenuation of disturbance and initial-state uncertainties of controllers and derive a necessary and sufficient condition for the solution.

Then, an $\mathcal{H}_\infty$ control problem which considers a initial state uncertainties of both of the plant and the controller was formulated and solved. The obtained three problems and their conditions are summarized in Table I. The obtained result is a natural extension of the previous result and it looks reasonable.

Furthermore, we applied a disturbance and initial state uncertainties attenuation control problem to the magnetic suspension system, and showed the property and effectiveness of the proposed mixed attenuation controller by some control simulation results.

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