

# $H_\infty/\mu$ DIA Control of the Magnetic Suspension System

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**Abstract**—This paper deals with  $H_\infty/\mu$  DIA control of the magnetic suspension system.  $H_\infty$  DIA control is an  $H_\infty$  control problem which treats a mixed Disturbance and an Initial-state uncertainty Attenuation(DIA) and supplies  $H_\infty$  controls with good transients.  $H_\infty$  DIA controller has a good initial response property, however its robust performance might be improved. We propose  $H_\infty/\mu$  DIA Control which is to find a multi-objective controller to achieve both the  $H_\infty$  DIA condition for good initial responses/transient responses and the structured singular value  $\mu$  condition for robust performance. We apply this proposed approach to the magnetic suspension system, and design a robust controller which has both good properties. Finally experimental results show effectiveness of the proposed control system design framework.

## I. INTRODUCTION

Conventional  $H_\infty$  control attenuates the only effect of disturbances on controlled outputs and is originally defined under the assumption that the initial states of the system are zero. If the initial states are non-zero, some transients properties with the system applying an  $H_\infty$  control will deteriorate. We proposed an  $H_\infty$  control which achieves a mixed Disturbance and Initial-state uncertainty Attenuation in controlled outputs[1]. This mixed attenuation  $H_\infty$  control ( $H_\infty$  DIA Control) has good initial/transient response properties[2], however its robust performance might be improved. There are some previous works on studying about control performance of transient property and robust performance[3], [4]. Yang et. al.[3] utilized adaptive robust nonlinear control and Uchiyama et. al.[4] applied 2-degree of freedom control with  $\mu$ -synthesis. Both results are effective, but the only problem here is a complexity of their implementations and a fragileness to initial state uncertainties.

To achieve the good transient property and robust performance under the initial state uncertainties of the plant, we apply a  $D$ - $K$  iteration technique[5] for improving robust performance to  $H_\infty$  DIA control. Here,  $H_\infty/\mu$  DIA Control is to find a multi-objective controller to achieve  $H_\infty$  DIA condition for good initial responses/transient responses and the structured singular value  $\mu$  condition for robust performance[6].

This proposed approach is applied to the magnetic suspension system[7] and its effectiveness is evaluated via some control experiments. Magnetic suspension systems can suspend a magnetic body by magnetic force without any contact[7], which requires feedback control in order

to be workable. Recently, magnetic suspension systems including active magnetic bearings[8] and also magnetic field control[9] seem to be one of the hot topics in control application field. Nonlinear control approaches are recently focused in this field[3], [8], [9], but our approach taken here is a reliable linear robust control methodology[1], [5].

Finally, compared with the conventional  $H_\infty$  DIA controller, usefulness and effectiveness of the proposed  $H_\infty/\mu$  DIA control design framework considering initial-state uncertainty will be shown via some experimental results for transient responses and for improving robust performance.

## II. $H_\infty/\mu$ CONTROL PROBLEM

Consider the linear time-invariant system which is defined on the time interval  $[0, \infty)$  and described by

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u, & x(0) &= x_0 \\ z &= C_1x + D_{12}u \\ y &= C_2x + D_{21}w \end{aligned} \quad (1)$$

where  $x \in R^n$  is the state and  $x_0$  is the unknown initial state;  $u \in R^r$  is the control input;  $y \in R^m$  is the observed output;  $z \in R^q$  is the controlled output;  $w \in R^p$  is the disturbance. Without loss of generality, we regard  $x_0$  as the initial-state uncertainty, and  $x_0$  as a known initial-state case. The disturbance  $w(t)$  is a square integrable function defined on  $[0, \infty)$ .  $A$ ,  $B_1$ ,  $B_2$ ,  $C_1$ ,  $C_2$ ,  $D_{12}$  and  $D_{21}$  are constant matrices of appropriate dimensions and satisfies that

- $(A, B_1)$  is stabilizable and  $(A, C_1)$  is detectable
- $(A, B_2)$  is controllable and  $(A, C_2)$  is observable
- $D_{12}^T D_{12} \in R^{r \times r}$  is nonsingular
- $D_{21} D_{21}^T \in R^{m \times m}$  is nonsingular

For system (1), every admissible control  $u(t)$  is given by a linear time-invariant system of the form

$$\begin{aligned} u &= J\zeta + Ky \\ \dot{\zeta} &= G\zeta + Hy, & \zeta(0) &= 0 \end{aligned} \quad (2)$$

which makes the closed-loop system given by (1) and (2) internally stable, where  $\zeta(t)$  is the state of the controller of a finite dimension;  $J$ ,  $K$ ,  $G$  and  $H$  are constant matrices of appropriate dimensions.

For the system and the class of admissible controls described above, consider the  $H_\infty/\mu$  DIA control problem to consider both of transient response and robust performance under the initial state uncertainty of the plant.

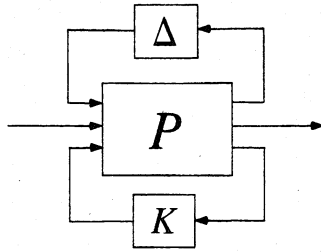


Fig. 1. Feedback System with Uncertainty

### A. $H_\infty$ DIA Control Problem

Find an admissible control attenuating disturbances and initial state uncertainties in the way that, for given  $N > 0$ ,  $z$  satisfies

$$\|z\|_2^2 < \|w\|_2^2 + x_0^T N^{-1} x_0 \quad (3)$$

for all  $w \in L^2[0, \infty)$  and all  $x_0 \in \mathcal{R}^n$ , s.t.,  $(w, x_0) \neq 0$ .

We call such an admissible control the **Disturbance and Initial state uncertainty Attenuation (DIA)** control. The weighting matrix  $N$  is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation[1].

### B. $\mu$ -Synthesis Problem

LFT and  $\mu$ -synthesis[5], [6] have come to play an important role in control system design and provide a uniform framework for realization, analysis and synthesis for uncertain systems in Fig.1, where  $\Delta$  is a structured uncertainty and  $P$  is a generalized plant and  $K$  is a controller.

The block structure  $\Delta$  is generally defined as

$$\Delta = \text{diag}[\delta_1 I_{r_1}, \dots, \delta_S I_{r_S}, \Delta_1, \dots, \Delta_F] \quad (4)$$

$$: \delta_i \in \mathcal{R}, \Delta_j \in \mathcal{C}^{m_j \times m_j}$$

For consistency among all the dimensions, we must have

$$\sum_{i=1}^S r_i + \sum_{j=1}^F m_j = n. \quad (5)$$

Here it is well known that the structured singular value  $\mu_\Delta(M)$  is defined for matrices  $M \in \mathcal{C}^{n \times n}$  with the block structure  $\Delta$  as

$$\mu_\Delta(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0\}} \quad (6)$$

unless no  $\Delta \in \Delta$  makes  $(I - M\Delta)$  singular, in which case  $\mu_\Delta(M) := 0$ [5]. Then, the control problem is to find the controller  $K(s)$  which achieves the following robust performance condition.

$$\sup_{\omega \in \mathcal{R}} \mu_\Delta[F_1(P(j\omega), K(j\omega))] < 1 \quad (7)$$

### C. $H_\infty/\mu$ DIA Control Problem

Our final control problem is to find an admissible controller to satisfy both  $H_\infty$  DIA control condition in (3) and robust performance  $\mu$  condition in (7).

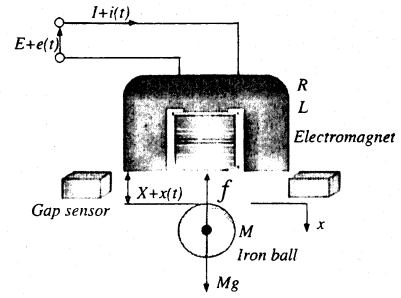


Fig. 2. Magnetic Suspension System

## III. SYSTEM DESCRIPTION AND MODELING

Consider the electromagnetic suspension system shown schematically in Fig.2. An electromagnet is located at the top of the experimental system. The control problem is to levitate the iron ball stably utilizing the electromagnetic force. The mass  $M$  of the iron ball is 286[g], and steady state gap  $X$  is 3[mm]. Note that this simple electromagnetic suspension system is unstable without feedback control. A standard optical gap sensor is placed both sides of the ball to detect the distance between the iron ball and the electromagnet.

Under some assumptions around the steady state operation[7], we can derive the following three equations, which show an equation of the motion of the iron ball (8), electromagnetic force (9) and equation of an electric circuit of the electromagnet(10) respectively.

$$M \frac{d^2 x(t)}{dt^2} = Mg - f + v_m(t) \quad (8)$$

$$f(t) = k \left( \frac{I + i(t)}{X + x(t) + x_0} \right)^2 \quad (9)$$

$$L \frac{di(t)}{dt} + R(I + i(t)) = E + e(t) + v_L(t) \quad (10)$$

where  $M$  is the mass of the iron ball,  $X$  is the steady gap between the electromagnet(EM) and the iron ball,  $x(t)$  is the deviation from  $X$ ,  $I$  is the steady current,  $i(t)$  is the deviation from  $I$ ,  $E$  is the steady voltage,  $e(t)$  is the deviation from  $E$ ,  $f(t)$  is the electromagnetic force,  $k$  and  $x_0$  are coefficients of  $f(t)$  which are determined by experiments,  $L$  is an inductance of the EM,  $R$  is a resistance of the EM, and  $v_m(t)$ ,  $v_L(t)$  are exogenous disturbance and uncertainties. The nominal model parameters of the plant are given in Table I.

In the case we apply the linear control theory with respect to this system and the problem is that the equation of the electromagnetic force (9) is nonlinear concerning  $x(t)$  and  $i(t)$ . Here we utilize the standard linearization approach based on the Taylor series expansion around the operating point.

$$f(t) = k \left( \frac{I}{X + x_0} \right)^2 - K_x x(t) + K_i i(t) \quad (11)$$

where  $K_x = 2kI^2/(X + x_0)^3$ ,  $K_i = 2kI/(X + x_0)^2$  !

TABLE I  
PHYSICAL MODEL PARAMETERS

Symbol	Parameter Name	Value	Unit
$M$	Mass of the ball	0.286	kg
$X$	Steady Gap	$3.000 \times 10^{-3}$	m
$I$	Steady Current	0.843	A
$E$	Steady Voltage	8.47	V
$k$	coefficient of $f$	$2.14 \times 10^{-4}$	$\text{Nm}^2/\text{A}^2$
$x_0$	coefficient of $f$	$4.36 \times 10^{-3}$	m
$R$	Resistance	9.50	$\Omega$
$L$	Inductance	0.300	H

The gap sensor provides the information for the gap  $x(t)$  a noise. Hence the measurement equation for  $y_g(t)$  can be written as

$$y_g(t) = x(t) + w_0(t) \quad (12)$$

where  $w_0(t)$  represents the sensor noise as well as the model uncertainties.

Moreover, the steady state equations are given by  $Mg = k \left( \frac{I}{X+x_0} \right)^2$  and  $RI = E$ , then summing up the above results, the state equations for the system are

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g u_g + D_g v_0 \\ y_g &= C_g x_g + w_0 \end{aligned} \quad (13)$$

where  $x_g := [x \ \dot{x} \ i]^T$ ,  $u_g := e$ ,  $v_0 := [v_m \ v_L]^T$ ,

$$\begin{aligned} A_g &= \begin{bmatrix} 0 & 1 & 0 \\ 2670 & 0 & -23.3 \\ 0 & 0 & -31.6 \end{bmatrix}, & B_g &= [0 \ 0 \ 3.33]^T \\ C_g &= [1 \ 0 \ 0], & D_g &= \begin{bmatrix} 0 & 0 \\ 3.50 & 0 \\ 0 & 3.33 \end{bmatrix}. \end{aligned}$$

Here  $(A_g, B_g)$  and  $(A_g, D_g)$  are controllable, and  $(A_g, C_g)$  is observable.

#### IV. CONTROL SYSTEM DESIGN

In this section, we apply the  $H_\infty/\mu$  DIA control to the magnetic suspension system and design a control system.

##### A. Construction of the generalized plant

First let us consider the system disturbance  $v_0$ . Since  $v_0$  mainly acts on the plant in a low frequency range in practice, it is helpful to introduce a frequency weighting factor. Hence let  $v_0$  be of the form

$$v_0 = W_v(s) w_2 \quad (14)$$

$$W_v = \Phi C_v (sI - A_v)^{-1} B_v, \quad \Phi = [1 \ 1]^T \quad (15)$$

where  $W_v(s)$  is a frequency weighting whose gain is relatively large in a low frequency range, and  $w_2$  is a (1, 2) element of  $w$ . Consider the system disturbance  $w_0$  for the output. The disturbance  $w_0$  shows an uncertain influence caused via unmodeled dynamics, and define

$$w_0 = W_w w_1 \quad (16)$$

where  $W_w$  is a weighting scalar, and  $w_1$  is a (1, 1) element of  $w$ . Note that  $W_w$  is sometimes frequency dependent, but it is selected as a scalar for the sake of simplicity.

Next we consider the variables which we want to regulate. In this case, since our main concern is in the stabilization of the iron ball, the gap  $x(t)$  and the corresponding velocity  $\dot{x}(t)$  are chosen; i.e.,

$$z_g = F_g x_g, \quad F_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (17)$$

Then, as the error vector, let us define as follows

$$z_2 = \Theta z_g, \quad \Theta = \text{diag} [\theta_1 \ \theta_2] \quad (18)$$

where  $\Theta$  is a weighting matrix on the regulated variables  $z_g$ , and  $z_2$  is a (1, 2) element of  $z$ . This value  $\Theta$ , as yet unspecified, are also free design parameters.

Furthermore the control input  $u$  should be also regulated, and we define

$$z_1 = \rho u_g \quad (19)$$

where  $\rho$  is a weighting scalar, and  $z_2$  is a (1, 2) element of  $z$ . Finally, let  $x := [x_g^T \ x_v^T]^T$ , where  $x_v$  denotes the state of the frequency weighting  $W_v(s)$ , and  $w := [w_1^T \ w_2^T]^T$ ,  $z := [z_1^T \ z_2^T]^T$ , then we can construct the generalized plant as in the following:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} u \end{aligned} \quad (20)$$

$$\begin{aligned} A &= \begin{bmatrix} A_g & D_g C_v \\ 0 & A_v \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 & D_g D_v \\ 0 & B_v \end{bmatrix}, \\ B_2 &= \begin{bmatrix} B_g \\ 0 \end{bmatrix}, & C_1 &= \begin{bmatrix} 0 & 0 \\ \Theta F_g & 0 \end{bmatrix}, & D_{12} &= \begin{bmatrix} \rho \\ 0 \end{bmatrix}, \\ C_2 &= [C_g \ 0], & D_{21} &= [W_w \ 0] \end{aligned}$$

##### B. Problem Setup for Control System Design

Next, for the robust performance synthesis, we define the definite block structure  $\Delta$  in this system as follows.

$$\Delta := \left\{ \begin{bmatrix} \Delta_w & 0 \\ 0 & \Delta_v \end{bmatrix} : \Delta_w \in \mathcal{C}, \Delta_v^{1 \times 2} \in \mathcal{C}, \forall \omega \right\} \quad (21)$$

where  $\Delta_w$  is an additive perturbation including parametric uncertainty, linearization error and unmodeled dynamics; and  $\Delta_v$  is defined as  $\Delta_v = [\Delta_{v1} \ \Delta_{v2}]$  and is a fictitious uncertainty for control performance. The final interconnection structure with an unspecified controller  $K$  by LFT representation in Fig.3.

Next our control problem is defined by using the interconnection structure in Fig.3 as:

**Control Problem Setup:** Find an admissible controller  $K(s)$  that achieves both of the DIA condition in (3) and the structured singular value  $\mu$  condition (7) for the interconnection structure. We call such an admissible controller  $K(s)$   $\mu$ -DIA Controller.

##### C. Design Procedure of the $\mu$ -DIA Controller

We design the  $\mu$ -DIA controller based on the following Nine-Step procedure. Iterative calculations concerning to design parameters and  $D$ -scaling matrix are done to obtain appropriate numerical sets on MATLAB, then we obtain a numerical  $\mu$ -DIA controller  $K(s)$  directly.

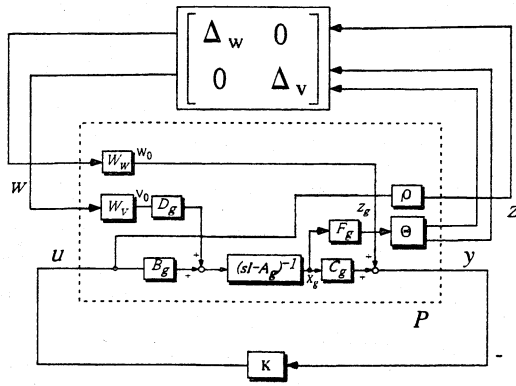


Fig. 3. Generalized Plant with Uncertainty

**[Step 1] Select a weighting function  $W_v(s)$ :**

$W_v(s)$  is a frequency weighting function which its gain is relatively large in a low frequency range. This parameter is mutually related to a low gain of the controller  $K$  and the controller performance.

**[Step 2] Select a weighting function  $W_w(s)$ :**

$W_w(s)$  is a frequency weighting function and this is related to robustness. Bigger choice of  $W_w$  could mean allowing bigger uncertainties. Here we selected  $W_w$  as a scalar for simplicity, but it can be chosen as a frequency function.

**[Step 3] Select a weighting matrix  $\Theta$ :**

$\Theta$  is a weighting matrix on the regulated variables  $z_g$  which means that  $\theta_1$  and  $\theta_2$  regulate  $x(t)$  and  $\dot{x}(t)$  in  $x_g(t)$  respectively.

**[Step 4] Select a weighting scalar  $\rho$ :**

$\rho$  is a weighting scalar on the input variable  $u$  and  $\rho$  regulates the input  $u(t)$ .

**[Step 5] Construct the generalized plant and an  $H_\infty$  DIA controller:**

With a specified set of design parameters from [Step 1] to [Step 4], a generalized plant is constructed. The DIA controller is designed for this plant, and its state-space description is given by easy algebraic calculation.

**[Step 6]  $\mu$ -Analysis:**

Calculate  $\mu$  for  $F_l(P_i, K_i)$  and the block structure  $\Delta$ . Next we can get the scaling matrix  $\hat{D}_{i+1}(j\omega)$  to minimize the following function on every frequency  $\omega$

$$\bar{\sigma}[\hat{D}_{i+1}(j\omega)F_l(P_i, K_i)(j\omega)\hat{D}_{i+1}^{-1}(j\omega)] \quad (22)$$

Then, evaluate the condition;

$$\sup_{\omega \in \mathcal{R}} \bar{\sigma}[\hat{D}_{i+1}(j\omega)F_l(P_i, K_i)(j\omega)\hat{D}_{i+1}^{-1}(j\omega)] < 1 \quad (23)$$

If the condition (23) is achieved, then this procedure is completed and stopped. Otherwise go to the next step.

**[Step 7] Calculate the maximum matrix  $N$ :** Calculating the maximum  $N$  satisfies the condition (3). For the sake of simplicity, the structure of the matrix  $N$  is limited as

$$N = nI \quad (24)$$

where  $n$  is a positive scalar number and  $I$  is a unit matrix of appropriate dimensions. This limitation on the positive

definite matrix  $N$  is for easy evaluation after the DIA analysis.

**[Step 8] Fix the scaling matrix  $D(s)$ :**

The scaling matrix  $\hat{D}_{i+1}(j\omega)D_i(j\omega)$  pointwise across frequency is transformed to the real rational matrix function  $D_{i+1}(s)$ . This step can be done by graphical matching using lower-order transfer functions.

**[Step 9] Reconstruct the generalized plant:**

Construct a new state-space model for the new generalized plant

$$P_{i+1} = \begin{bmatrix} D_{i+1} & 0 \\ 0 & 1 \end{bmatrix} P \begin{bmatrix} D_{i+1}^{-1} & 0 \\ 0 & 1 \end{bmatrix} \quad (25)$$

and return to [Step 5] and repeat the procedure until the controller  $K$  to achieve the condition is obtained.

**D. Design of  $\mu$ -DIA Controller**

After some iteration in MATLAB environment, these parameters are chosen by the above 9-step design procedure as follows;

$$\begin{aligned} W_v(s) &= \frac{5.0 \times 10^4}{s + 0.010}, & W_w &= 0.3 \\ \Theta &= \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.00010 \end{bmatrix} \\ \rho &= 4.0 \times 10^{-7} \end{aligned} \quad (26)$$

We obtained a following controller  $K(s)$  after the 2nd  $D$ - $K$  iteration, where the peak value of  $\mu_\Delta[F_l(P, K)]$  is 0.743 and a constant scaling matrix  $D$  is employed.

$$\begin{aligned} K(s) &= \frac{8.496 \times 10^8 (s + (48.68 + 20.39i))}{(s + (330.59 + 655.7i))} \\ &\times \frac{(s + (48.68 - 20.39i))(s + 7.1955)}{(s + (330.59 - 655.7i))(s + 811.19)(s + 0.01)} \end{aligned} \quad (27)$$

The maximum value of the weighting matrix  $N$  in (3) is given by

$$N = 4.561157 \times 10^{-3} \times I_4. \quad (28)$$

Calculated upper and lower bounds of  $\mu_\Delta[F_l(P, K)]$  and  $\bar{\sigma}(DF_l(P, K)D^{-1})$  with the controller  $K(s)$  in (28) are shown in Fig.4, where two solid lines show upper and lower bounds of  $\mu$  and the dashed line shows the maximum singular value respectively.

Since the peak value if the upper bound of  $\mu$  is less than 1 in Fig.4, the closed-loop system with uncertainties achieves the robust performance condition (7) and also achieve the "Control Problem Setup" condition for  $N = 4.561157 \times 10^{-3} \times I_4$ .

The frequency responses of the controller  $\mu$ -DIA controller and the conventional  $H_\infty$  DIA controller are shown in Fig. 5 by a solid line and a dashed line respectively. Fig.5 shows that both controllers have high gain at the low frequency and good roll-off property at high frequency range. These two controllers are obtained by using the same set of design parameters (26).