# FIM Analysis for EKF-based SLAM with Intermittent Measurements

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**Abstract–** The focus of study is to discuss a statistical behavior of FIM in each EKF update and determine its potential in providing sufficient information about Robotic Localization and Mapping problem with intermittent measurements. We provide a theoretical analysis result and prove that the FIM can successfully describe both upper and lower bounds for the state covariance matrix whenever measurement data is not arrived during robot observations. This approach can give a better picture on how information are processed in EKF when measurement data is partially unavailable. Simulation evaluations are included to verify the results and consistently demonstrate the expected outcome.

**Key Words:** EKF, Estimation, SLAM, Intermittent Measurements

## 1 Introduction

The robotic localization and mapping problem or alternatively known as *Simultaneous Localization and Mapping*(SLAM) problem[1,2] has been one of the fascinating themes in robotic research. SLAM demonstrates a condition of a robot or multi-robots which attempt to localize itself or themselves in an unknown environment while at the same time incrementally building a knowledge about its surroundings. These information are expressed in different kinds of ways, which are then used to achieve several task in diverse environments such as in mining, space exploration, or in hazardous area. See Fig.1 for details illustration about the SLAM problem.

Generally, most of the approaches in SLAM can be divided into two techniques, which are the parametric and non-parametric methods. Some of parametric approaches has been proposed such as the Extended Kalman Filter(EKF), Unscented Kalman Filter(UKF), and $H_\infty$ Filter(HF)[3,4]. In the other hand, Histogram Filter, and Particle Filter are those methods which representing the non-parametric techniques.

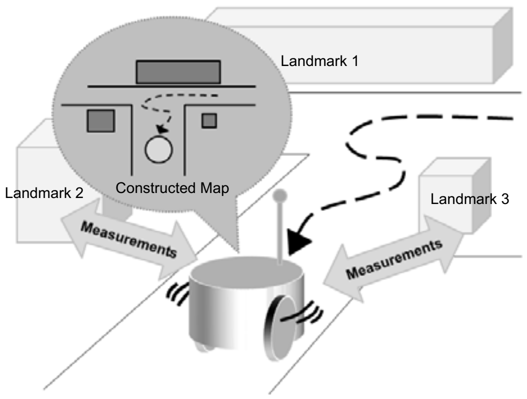


Fig. 1: SLAM problem

EKF-SLAM consistency and convergence properties have been discussed by some literatures[1,2]. They proved that the state covariance is monotonically decreased for both stationary and moving robot cases. The EKF inconsistency was also explained to describe the source of the problem. Related to this, CRLB[12] is one of the available approaches used to demonstrate consistency. Z.Jiang et.al[6] carried a CRLB evaluation for EKF-SLAM to determine the estimation behavior by considering several conditions. Andrea[6] studied the general SLAM accuracy with a known map by analyzing both process and measurement models using Fisher Information Matrix(FIM).

A case of EKF-SLAM with intermittent measurement is discussed in this paper. Based on Bernoulli process[9,10], it is possible to gain information about the estimation whenever measurement data is not available for some time interval. In probabilistic, these information are accessible through the system state error covariance. Until now, the intermittent measurements studies are mainly focused on linear and networks packet drops. Sinopoli et.al[9] claimed that there exist upper and lower bounds of state error covariance and their results has immense the research of intermittent measurement. S.Kluge reported that the estimation error will not be bounded if the initial state covariance, process and measurement error are too big under some relaxation of EKF assumptions. We show analytically the effect of these variables in this paper. In addition, the investigation of intermittent measurements are very limited for robotic system. One of them was demonstrated by Payeur[15]. He combines information from Jacobian transformation. Then by utilizing occupancy grid approach, he explains the condition when measurement data is partially loss. A scanning strategy also has been proposed to overcome such a situation in EKF-SLAM to occupy the system with an appropriate information[16]. However, none of them have reveals the theoretical explanation underneath. With regards to these papers, we propose the analysis using FIM[12] to provide a clearer notation in pursuing a SLAM problem whenever measurement data is unavailable.

In this paper, we derive the upper and lower bounds of the updated state error covariance by using FIM during intermittent measurement. We have found that based on FIM, the information is still available during intermittent measurement whether the measurement data is lost for a shorter time or longer time. Concurrently with the estimation, the upper and lower bounds about the state error covariance are also obtained. The updated state error covariance never surpassed the given bounds whether the measurement data is lost whether only for one sampling time or more. We also theoretically show that the uncertainties are gradually increasing when measurement data is unavailable. Based on the simulation results, the robot only has its confidence about the estimation when measurement data is available. As a result, FIM could be an alternative techniques to define the system upper and lower bounds when intermittent measurement is happen. Additionally, we guarantee that CRLB can be evaluated for SLAM problem[13,14].

## 2 EKF-Based SLAM

The SLAM problem can be described by process and measurement models. The process model describes the kinematic movement of the robot while the measurement model defines the behavior of sensors measurement when

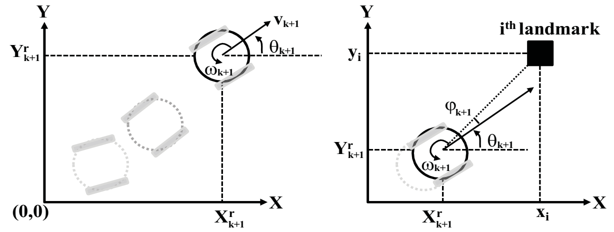


Fig. 2: Process model(left) and measurement model(right) of mobile robot localization and mapping problem

robot moving through the unknown environment. These two models are shown separately in Fig.2.

For process model, we consider a nonlinear discrete-time dynamical model as follows.

$$\theta_{k+1} = \theta_k + f_\theta(\omega_k, v_k, \delta\omega, \delta v) \quad (1)$$
$$X^r_{k+1} = X^r_k + (v_k + \delta v)T\cos[\theta_k] \quad (2)$$
$$Y^r_{k+1} = Y^r_k + (v_k + \delta v)T\sin[\theta_k] \quad (3)$$
$$L_{k+1} = L_k \quad (4)$$

where the robot states$\in \mathbb{R}^3$ are represented by the mobile robot pose angle $\theta_k$, and $X^r_k, Y^r_k$ are the $x,y$ cartesian coordinate of the mobile robot. While, $L^i_k \in \mathbb{R}^{2m}, m = 1,2,...,N$ is each respective landmark location in $x_i, y_i$ coordinate frame. Robot turning rate is defined by $\omega_k$ and its velocity by $v_k$. $\delta\omega, \delta v$ are the associated process noise to the mobile robot turning rate and its velocity respectively. $T$ is the sampling rate. The process model for the landmarks is unchanged as the landmarks are assumed to be stationary. Eqs.(1)-(4) are now represented by $X_{k+1}$ as an augmented state.

Based on process model, the robot motions are predictable and we can calculate the robot position at any time by using any robot proprioceptive sensors such as the encoder. However, are the calculations refer perfectly to the robot actual location? In this perspective, probabilistic SLAM provides a level of certainty about the estimation. In each robot motions, probabilistic method considers about the disturbances due to robot wheel misalignment and slippage by incorporating the analysis of state error covariance. The state error covariance determines the uncertainties of the system. Essentially, probabilistic SLAM is divided into two parts; prediction and update stage to comprehend about the system. This is shown as follow. As we applying EKF-SLAM algorithm, the prediction process is stated by

$$\hat{X}^-_{k+1} = f(\hat{X}_k, \omega_k, v_k, 0, 0) \quad (5)$$

There are no process noise included in the prediction such that $\delta\omega = 0, \delta v = 0$ and the initial robot velocity and its angular acceleration are given. $\hat{X}^-_{k+1}$ is the estimated augmented mobile robot and landmarks state with its associated covariance $P_{k+1} \in \mathbb{R}^{3+2m}$ and it is shown by the following equation.

$$P^-_{k+1} = f_r P_k f_r^T + f_{\omega v}\Sigma_k f_{\omega v}^T \quad (6)$$

Here $f_r$ is the Jacobian evaluated from the mobile robot motion in (1)-(3), and $\Sigma_k$ is the control noise covariance. $P_k$ is the previous state error covariance. For $T = 1$, the Jacobian for the process model yield the following expression.

$$f_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v\sin\theta & 1 & 0 & 0 \\ v\cos\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, f_{\omega v} = \begin{bmatrix} g_{\omega v} \\ 0 \end{bmatrix} \quad (7)$$

where $f_{\omega v}$ is the linearized process noise. We assume no process noise for landmarks. Therefore the linearized process noise for robot motion is $g_{\omega v}$. $I_n$ is an identity matrix with an appropriate dimension.

The mobile robot then makes the observations about it surroundings using its exteroceptive sensor and the behavior is shown by the following equations.

$$\begin{aligned} z_{i_{k+1}} &= \gamma_{k+1}\begin{bmatrix} r_i \\ \phi_i \end{bmatrix} \\ &= \gamma_{k+1}\begin{bmatrix} \sqrt{(x_i - X^r_{k+1})^2 + (y_i - Y^r_{k+1})^2} + v_{r_i} \\ \arctan\frac{y_i - Y^r_{k+1}}{x_i - X^r_{k+1}} - \theta_{k+1} + v_{\theta_i} \end{bmatrix} \end{aligned} \quad (8)$$

Equation (8) is then linearized and represented by

$$z_{i_{k+1}} = \gamma_{k+1} H_i X_{k+1} + v_{r_i\theta_i} \quad (9)$$

where $r_i$ and $\phi_i$ are the relative distance and angle between robot and any observable landmark. Above equation defines that the mobile robot keeps measuring relative distance and angle between itself and any $i^{th}$ landmark with some associated noises of $v_{r_i}$, $v_{\theta_i}$. Note that we simplify these noises by $v_{r_i\theta_i}$ in (9). Furthermore, $\gamma_{k+1}$ explains the stochastic behavior of measurement data whether it is available or not for a period of time. This variable relies on the Bernoulli process and has the following properties.

$$\begin{aligned} Pr\{\gamma_{k+1} = 1\} &= p \\ Pr\{\gamma_{k+1} = 0\} &= 1 - p \\ E[\gamma_{k+1}] &= E[\gamma^2_{k+1}] = p \end{aligned}$$

The mobile robot measurements can be represented by using Jacobian as mentioned by the following equation.

$$H_i = \begin{bmatrix} 0 & -\frac{dx_k}{r} & -\frac{dy_k}{r} & \frac{dx_k}{r} & \frac{dy_k}{r} \\ -1 & \frac{dy_k}{r^2} & -\frac{dx_k}{r^2} & -\frac{dy_k}{r^2} & \frac{dx_k}{r^2} \end{bmatrix} \quad (10)$$

where $r = \sqrt{(x_i - X^r_{k+1})^2 + (y_i - Y^r_{k+1})^2}$, $dx_k = x_i - X^r_{k+1}$ and $dy_k = y_i - Y^r_{k+1}$. Same as the process model, again the state error covariance is analyzed to obtain the efficiency about the estimation after measurement. The updated state error covariance $P^+_{k+1}$ is represented by below equation.

$$P^+_{k+1} = P^-_{k+1} - \gamma_{k+1} K_{k+1} \nabla H_i P^-_{k+1} \quad (11)$$

where $K_{k+1} = P^-_{k+1} H_i^T (H_i P^-_{k+1} H_i{}^T + R_{k+1})^{-1}$. Using these information, the corrected state update $\hat{X}^+_{k+1}$ is represented by

$$\hat{X}^+_{k+1} = f_r \hat{X}^-_{k+1} + \gamma_{k+1} K_{k+1}(H_i X_{k+1} - H_i \hat{X}^-_{k+1}) \quad (12)$$

Both of these models are then going through prediction and update recursively as long as the robot keep observing its surroundings. In this paper, we are concern to look into the uncertainties behavior whenever intermittent measurement occurs in SLAM. Thereby, we assume that the data association are perfectly given and the robot is in a planar environment.

In addition, the same characteristics about above measurement characteristics during intermittent measurement was also obtained by previous results[9, 10, 11]. The measurement innovation defines that whenever measurement data is unavailable, then the estimation is based on the following result.

$$H_i(X_{k+1} - \hat{X}^-_{k+1}) = \gamma_{k+1} A_{k+1}(C_{m_{k+1}} - V_{k+1}) \quad (13)$$

where $C^i_{m_{k+1}}$ and $V_{k+1}$ show the landmarks $x_i, y_i$ and robot $X^r_k, Y^r_k$ location respectively. $A_{k+1}$ is the linearized measurement matrix and is included in Eq.(14) later. Above equation portrays the resulting characteristics of the measurement model and agrees that $\gamma$ shows the statistical bound of the measurement model. $A_{k+1}$ is the Jacobian for measurement at point $A$ and is shown by

$$A = \begin{bmatrix} \frac{dx_A}{r_A} & \frac{dy_A}{r_A} \\ -\frac{dy_A}{r_A^2} & \frac{dx_A}{r_A^2} \end{bmatrix}, \quad dx_A = [x_i - x_A] \quad (14)$$

$$dy_A = [y_i - y_A], \quad r_A = \sqrt{dx_A^2 + dy_A^2} \quad (15)$$

### 2.1 Fisher Information Matrix

The FIM which is the inverse of CRLB[11, 13] emphasizes that the covariance matrix $P_k$ of an unbiased state estimator $\hat{x}_k$ has a lower bound and is given by

$$P_{k+1} = \mathbb{E}[(X_{k+1} - \hat{X}^-_{k+1})(X_{k+1} - \hat{X}^-_{k+1})^T] \geq J^{-1}_{k+1} \quad (16)$$

$J_k$ is the Fisher Information Matrix (FIM) and hold an equation as stated below.

$$J_{k+1} = D^{22}_{k+1} - D^{21}_{k+1}(J_{k+1} + D^{11}_{k+1})^{-1} D^{12}_{k+1} \quad (17)$$

In a nonlinear case, each element of the above expression is specified by the following equations.

$$\begin{aligned} D^{11}_{k+1} &= f_r^T Q_k^{-1} f_r \\ D^{12}_{k+1} &= -f_r^T Q_k^{-1} = [D^{21}_{k+1}]^T \\ D^{22}_{k+1} &= Q_k^{-1} + H_i^T R^{-1}_{k+1} H_i \end{aligned}$$

where $Q_k = f_{\omega v} \Sigma_{k+1} f^T_{\omega v}$, and $f_r$ defined in (7), and $H_k$ is already defined in (10). Further substitution of above elements to (14) and at the same time by utilizing Matrix Inversion Lemma, (14) yields below expression.

$$J_{k+1} = (f_r J^{-1}_{k+1} f_r^T + Q_k)^{-1} + H_i^T R^{-1}_{k+1} H_i \quad (18)$$

Under this condition, if a filter achieves the condition (16), then the filter is said to be efficient for estimation. For a given system which posses an initial state covariance $P_0$, the initial FIM hold the following property, $J_0 = P_0^{-1}$.

## 3 FIM Statistical Bound for SLAM

Following preparations are made to investigate the EKF-based SLAM efficiency using CRLB. Our paper aids the analysis for the literatures such as Z.Jiang et.al[6]. Nevertheless, we refine the uncertainties bound for SLAM under FIM representation. These information should assist better interpretation whenever a measurement data is missing. To give a better picture of measurement model, for a mobile robot observing a landmark at point $A$, the Jacobian matrix is given by

$$H_A = [-e \quad -A \quad A], \quad e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (19)$$

and the definitions for other elements in $A$ has been given. These variables have same meanings with respect to (10) and regarding to the observation at a specific point.

**Assumption 1** *Both of the process and measurement noises holds the following characteristics.*

$$\mathbb{E}\left( \begin{bmatrix} w_{k+1} & 0 \\ 0 & v_{k+1} \end{bmatrix} \begin{bmatrix} w_{k+1} & 0 \\ 0 & v_{k+1} \end{bmatrix}^T \right) = \begin{bmatrix} Q_{k+1} & 0 \\ 0 & R_{k+1} \end{bmatrix}$$

*where $w_{k+1}$ is the process noise variance and $v_{k+1}$ is the measurement noise variance. $Q_{k+1} \geq 0$ and $R_{k+1} > 0$ are the process and measurement noise covariances respectively.*

By utilizing (16), the convergence behavior of EKF state error covariance must satisfy the following order.

$$P_k > P_{k+1} > \cdots > P_n$$

Analogously, this means that $P_k > P_{k+1} \geq J^{-1}_{k+1}$. This property also explains that FIM can be used to describe the lower bound of the state error covariance. Furthermore, FIM can be employed to acquire the upper bound of state error covariance. Hence, FIM sufficiently acts as a tool to evaluate the whole system whenever measurement data is intermittently missing during robot observations about it surroundings. It also can demonstrate the system uncertainties at each respective update.

We now present the FIM analysis whenever measurement data is not arrived. In our approach, we analyze FIM behavior in each estimation to obtain the upper and lower bound of the state error covariance. The lower bound is actually described by CRLB which utilizes FIM to demonstrate a minimum level that a state error covariance of a filter can achieved. Based on (16) and aforementioned definition of intermittent measurement, the FIM now yield the following expression.

$$J_{k+1} = (f_r J_k^{-1} f_r^T + Q_k)^{-1} + \gamma_{k+1} H_i^T R^{-1}_{k+1} H_i \quad (20)$$

where $\gamma_{k+1}$ described the stochastic behavior of measurement data arrival at time $k+1$.

To visualize more about above expression, consider a stationary robot observing a landmark at a point for $n-$times observations. In this case, the FIM for $n-$times observations $J^n_{k+1}$ is represented by the following equation.

$$J^n_{k+1} = (f_r J_k^{-1} f_r^T + nQ_k)^{-1} + n\gamma_{k+1} H_i^T R^{-1}_{k+1} H_i \quad (21)$$

Stated above, it can be concluded that the measurement update is very important to the system. If more observations are being made by the robot without any lost of measurement data, then the state error covariance will exhibit smaller uncertainties as more information are available.

**Remark 1** *Note that observing only a single landmark $n-$times is insufficient for the robot to localize itself in an unknown environment. However, according to S.Huang et.al[2], this characteristic is important to understand how the estimation is done and in what manner does the measurement data can improved the estimation at each observation.*

**Proposition 1** *If the initial state covariance and both process and measurement noises are very big, the estimation has very big uncertainties whenever measurement data is not arrived. The condition become worse if the measurement data is not available for longer time such that*

$$\lim_{k \to \infty} J_\infty = (P_0 + Q_k)^{-1} \to 0 \qquad \forall k > 0$$

**Proof** The comparison test is used to evaluate the proposition. Assume that robot is stationary at point $A$ and starts observing its surroundings. If no measurement data is available, after one update and the next update we have

$$\begin{aligned} J_k &= (P_0 + Q_k)^{-1} \\ J_{k+1} &= (P_0 + Q_k + Q_{k+1})^{-1} \equiv (P_0 + 2Q_k)^{-1} \end{aligned}$$

Assume that process noise has almost same magnitude for each prediction as mentioned above. As $P_k = J_k^{-1}$, and $k \to \infty$, we represent above conditions as

$$\lim_{k\to\infty} J_\infty = (P_0+kQ_k)^{-1} < (P_0+kQ_k)^{-1} + H_i^T R_{k+1} H_i$$

As a result, the longer measurement data is unavailable, then the state error covariance is going to be $\infty$ which means the estimation continuously diverges. □

Therefore, this situation cannot guarantee a better result as initial state covariance, process and measurement noises still affects the estimation performance. This proposition implies that intermittent measurement in SLAM may lead to unfavorable circumstances about the estimation.

Look upon a case when a measurement data is not available at certain time $k(k=1,2,...)$. Notice that the FIM will refer to its previous information as no measurement data is arrived at a certain time to update the system. Motivated by this condition, we conduct a deterministic study to derived FIM lower and upper bounds for the system.

**Definition 1** *For an initial state covariance $P_0 > 0$, there exist a real random number $\rho_k > 0$ and $Q_k \geq 0$ for each EKF update such that*

$$\rho_k = f_r P_k f_r^T > 0 \tag{22}$$
$$J_k(\leq \rho_k) > 0 \tag{23}$$
$$Q_k = f_{\omega v}\Sigma_k f_{\omega v}^T \geq 0 \tag{24}$$

The first definition simply interprets that the state error covariance matrix always yields a positive definite matrix in each update. It is a main property to be analyzed in probabilistic SLAM. Equation (23) is very important to ensure that at least a solution exists during estimation. Lastly, (24) is a definition that the Jacobian of process noise is at least a positive semidefinite matrix at each time.

**Remark 2** *Suppose that $Q_k$ is a singular, $Q_k$ can be substituted by $Q_k + \varepsilon I$ for some very small positive $\varepsilon$[11]. Such a case is being considered in most SLAM problem, which assumes that there are almost no process noise for landmarks. By this setting, $Q_k$ becomes a non-singular matrix and therefore enabling us to examine the behavior and its effect in the case of intermittent measurements. We assume that in every process, the process noise is represented by*

$$\bar{Q}_k = f_{v\omega}\Sigma_k f_{v\omega}^T + \varepsilon I_n$$

*This equation holds in each respective robot movement unless otherwise stated.*

Applying the *Matrix Inversion Lemma* to the first term of the right hand (20), the following expression is given.

$$J_{k+1} = \rho_k^{-1} - \rho_k^{-1}\bar{Q}_k\rho_k^{-1} + \psi_k + \gamma_{k+1}H_i^T R_{k+1}^{-1} H_i \tag{25}$$

where $$\psi_k = \rho_k^{-1}\bar{Q}_k(\rho_k+\bar{Q}_k)^{-1}\bar{Q}_k\rho_k^{-1}$$

If the process noise is extremely small and can be neglected, then above result is same to S.Huang[2] especially for a case of a stationary robot observing a landmark at some point with an initial state covariance $P_0(\in \mathbb{R}^{3+2m}) > 0$. Based on their results, when process noise covariance $Q_k$ is so small and can be neglected, then EKF update and its convergence holds the following criteria.

$$J_k = P_0^{-1} + \gamma_{k+1}H_i^T R_{k+1}^{-1} H_i \tag{26}$$

$$J_{n\to\infty}^{-1} \leq \begin{bmatrix} P_0 & P_0 H_A^T A^{-T} \\ A^{-1}H_A P_0 & A^{-1}H_A P_0 A H_A^T A^{-T} \end{bmatrix} \tag{27}$$

Equation (27) generally defines that if the observations at a point are made successively, in the limit the state covariance is converging to the given equation. The assumption of $Q_k$ is small and can be ignored has made S.Huang et.al results a general conclusion about EKF-SLAM convergence properties. However, whenever the process noise covariance has to be considered, the process noise has a significant effect to the overall estimation. Even more, the process noise also defines the system state error covariance boundedness. We now move to investigate further about the contribution of these equations to a case of intermittent measurements.

Based on *Assumption 1* and *Definition 1*, it is understood that $Q_k \geq 0$. Besides, FIM must satisfy $J_k > 0$ to ensure that there exist a solution to EKF-SLAM. We denote at time $k$, the process noise is represented by either one of the following. $\bar{Q}_k$ to express that it is the maximum process noise covariance and $\underline{Q}_k$ for the minimum process noise covariance. These expression equivalently means that the process noise are not normally distributed and has either highest variance $\bar{Q}_k$ or lowest variance $\underline{Q}_k$.

**Lemma 1** *Given $P_0, Q_k, R_k > 0$. If a measurement data is missing in the interval of $1 < k < N(N > 0)$ time, then the FIM lower bound and upper bound are shown as follows.*

$$\underline{J}_{k+1} = \rho_k^{-1} + \rho_k^{-1}\bar{Q}_k(\rho_k+\bar{Q}_k)^{-1}\bar{Q}_k\rho_k^{-1} + \gamma_{k+1}H_k^T R_k^{-1} H_k \tag{28}$$

$$\bar{J}_{k+1} = \rho_k^{-1} - \rho_k^{-1}\underline{Q}_k\rho_k^{-1} + \gamma_{k+1}H_k^T R_k^{-1} H_k \tag{29}$$

**Proof** Equation (25) are applied to investigate the statistical bound of the state error covariance updates. Convergence results from S.Huang et.al[2] are also referred to evaluate the update.

Based on (25), we obtained a maximum and minimum value of FIM. Note that $J_k > 0$ must be satisfied at each update to ensure at least a solution exist. Besides, from S.Huang et.al[2], if the process noise is too small such that it can be neglected, then after recursive update, the estimation converges to the initial state covariance $P_0$. By making $\bar{Q}_k = 0$ in (25), we obtained that $\rho_k = P_0$(refer to [2] for explicit derivation). By this fact, for $P_0 > 0$ and when $\bar{Q}_k = 0$, then we can conclude that (25) holds the following property for $n-$times observations.

$$J_{n\to\infty} = P_0$$

This properties is preserved in all observations. We then have the following expression.

$$J_{k+1} > J_k > 0$$

Thus, the minimum FIM information can be given by the following equation(by means that (25) achieved its minimum information).

$$\underline{J}_{k+1} = \rho_k^{-1} + \rho_k^{-1}\bar{Q}_k(\rho_k+\bar{Q}_k)^{-1}\bar{Q}_k\rho_k^{-1} \tag{30}$$

The FIM upper bound $\bar{J}_{k+1}$ is given as

$$\bar{J}_{k+1} = \rho_k^{-1} - \rho_k^{-1}\underline{Q}_k\rho_k^{-1} \tag{31}$$

Both (30) and (31) contributes the upper and lower information bound for the EKF-SLAM with intermittent measurements. □

*Lemma 1* has described the FIM lower and upper bounds when measurement data is not available. By determining the possible maximum or minimum of information obtained during intermittent measurement, we are able to infer the updated state error covariance condition. These results are more deterministic than previous findings which helps designer to comprehend better information about the system(see [10]−[12] for further details). It seems normal that if when measurement data is missing then FIM acquired previous data to update its current information.

However, as we shown in above lemma, when measurement data is unavailable, FIM does not refer back to its previous data but is strictly bound to (28) or (29). Besides, process noise covariance characteristics, the updated state error covariance also depends on the following equation which has been stated earlier in this paper.

$$H_i(X_{k+1} - \hat{X}^-_{k+1}) = \gamma_{k+1} A_{k+1} (L_{m_{k+1}} - R_{k+1})$$

For EKF-SLAM, for any given $P_0 > 0$, the state error covariance $P_k$ is converging to $P_0$ after sufficient observations if and only if $\bar{Q}_k$ is so small and can be neglected[2]. Besides, it has been guaranteed that when the robot is moving, the convergence results is shown by the addition of $P_0$ and its associated process noise distribution.

Now we derive the statistical bound for the state error covariance $P_k$ whenever the measurement data is intermittently unavailable at $k+1$. A condition is also proposed to ensure that the state error covariance is converging. We show that if $\rho_{k+1} > \bar{Q}_{k+1}$ and $\bar{Q}_k$ is invertible, then the statistical bounds are exist. Even though the process noise such as the wheel misalignment and slippage do not obey normal distribution and is unknown, a designer is able to obtain the robot kinematics with probabilistic method under certain knowledge. If the process noise covariance is enormously bigger than the initial state covariance, then the prediction result in high uncertainties about the system. Consequently, the estimation become inconsistent and yield erroneous position estimations.

**Lemma 2** *Given $\rho_{k+1} \geq 0$. In EKF-SLAM, if no measurement data is available during robot observations, then the estimation is still possible if and only if $\rho_{k+1} > \bar{Q}_{k+1}$ such that if $\bar{Q}_{k+1} > P_0$, then the estimation is insufficient.*

**Proof** The proof can be easily obtained by analyzing the FIM. Referring to (18), when the stationary robot observes it surroundings for the first time and then moves, we have the following expression.

$$P_k^{-1} = P_0^{-1} + H_i^T R_k^{-1} H_i$$

By previous results[(2)], if more observations are made by the robot such that $n \longrightarrow \infty$, then $P_{n\to\infty} \longrightarrow P_0$. At the next stage of $k+1$, when the robot moves and due to slippage and other disturbance, we obtain the following.

$$P_{k+1}^{-1} = (\rho_k + \bar{Q}_k)^{-1} + H_i^T R_{k+1}^{-1} H_i$$

where $\rho_k$ is defined in (23). Based on above, if $\bar{Q}_k > \rho_k$ then it can be easily identified that the state error covariance is not converging to $P_0$. Instead, it converges to a bigger value than $P_0$ which depends to the process noise covariance. The result become worst if process noise is enormously bigger than $\rho_k$ and if intermittent measurement is occurred during observations, thus producing erroneous results about the state. If the process noise is keep increasing or the robot lost capability to sense it motions, then estimation is impossible in SLAM. □

**Theorem 1** *Assume that (24) is satisfied. Consider that the initial state covariance $P_0 > 0$ and Assumption 1 are satisfied. If a measurement data is not arrived at any $k, 1 < k < N$ time. Then if $\rho_{k+1} > \bar{Q}_{k+1}$ is achieved, the state error covariance $P_{k+1}$ is bounded as the following.*

$$\underline{P}_{k+1} \leq P_{k+1} \leq \bar{P}_{k+1} \tag{32}$$

*such that*

$$\underline{P}_{k+1} = \rho_k - \underline{Q}_k(\rho_k + \underline{Q}_k + \underline{Q}_k \rho^{-1} \underline{Q}_k)^{-1} \underline{Q}_k \tag{33}$$

$$\bar{P}_{k+1} = \rho_k + (\bar{Q}_k^{-1} - \rho_k^{-1})^{-1} \tag{34}$$

*In other words, the upper bound of state error covariance update $\bar{P}_{k+1}$ is shown by $\underline{J}_{k+1}^{-1}$, while the lower bound of state error covariance update $\underline{P}_{k+1}$ is presented by $\bar{J}_{k+1}^{-1}$. In a case of a stationary robot observing landmarks, if a measurement data is intermittently missing at $1 < k < N$, and process noise $Q_k$ is very small, then the upper bound is restricted and bounded to the amount of previous state error covariance $P_k$.*

**Proof** The proof is divided into two parts comprising about the upper and lower bounds of the state error covariance.

1. (**Lower bound for state error covariance**)
   We attempt to find the maximum value of $\bar{J}_k$ for a given initial state covariance $P_0 > 0$, transition matrix $f_r$ and measurement matrix $H_k$. Assume that *Assumption 1* is satisfied. In other words,
   $$\underset{P_0,\underline{Q}_k,R_k,f_r,H_k}{\arg\min}(\underline{P}_{k+1}) := \{\bar{J}_{k+1} | \forall P_0, Q_k, f_r > 0\}$$
   From Matrix Inversion Lemma, FIM lower bound $\bar{J}_{k+1}$ has its maximum information if the equation become as following.
   $$\begin{aligned}\bar{J}_{k+1} &= \rho_k^{-1} + \rho_k^{-1}\underline{Q}_k(\rho_k + \underline{Q}_k)^{-1}\underline{Q}_k\rho_k^{-1} \\ &\quad + \gamma_{k+1} H_k^T R_k^{-1} H_k\end{aligned}$$
   Now we can determine the state error covariance as follows. If the measurement data is missing at $k+1$-time, then after some arrangement and by Matrix Inversion Lemma, we finally obtained that the state error covariance yield the following expression.
   $$\begin{aligned}\underline{P}_{k+1} &= [\rho_k^{-1} + \rho_k^{-1}\underline{Q}_k(\rho_k + \underline{Q}_k)^{-1}\underline{Q}_k\rho_k^{-1}]^{-1} \\ &= \rho_k - \underline{Q}_k(\rho_k + \underline{Q}_k + \underline{Q}_k\rho^{-1}\underline{Q}_k)^{-1}\underline{Q}_k\end{aligned}$$
   The second term on the right hand side of above equation can be easily evaluated to ensure $\underline{P}_{k+1}$ always yields a positive definite.

2. (**Upper bound for state error covariance**)
   The lower bound of FIM that generates the upper bound of state error covariance is given by
   $$\underset{P_0,\bar{Q}_k,R_k,f_r,H_k}{\arg\max}(\bar{P}_{k+1}) := \{\underline{J}_{k+1} | \forall P_0, Q_k, f_r > 0\}$$
   With reference to (25), we suggest that the following equation describe the lower bound of FIM $\underline{J}_{k+1}$.
   $$\underline{J}_{k+1} = \rho_k^{-1} - \rho_k^{-1}\bar{Q}_k\rho_k^{-1} + \gamma_{k+1} H_k^T R_k^{-1} H_k$$
   If measurement data is not arrived at $k+1$, then
   $$\underline{J}_{k+1} = \rho_k^{-1} - \rho_k^{-1}\bar{Q}_k\rho_k^{-1}$$
   Determining the upper state error covariance apparently give us the following result.
   $$\begin{aligned}\bar{P}_{k+1} &= [\rho_k^{-1} - \rho_k^{-1}\bar{Q}_k\rho_k^{-1}]^{-1} \\ &= \rho_k + (\bar{Q}_k^{-1} - \rho_k^{-1})^{-1}\end{aligned}$$
   $\bar{q}_k^{-1}$ is a pseudo inverse of the process noise such that there are very small landmarks process noise induced in the system. The inverse term of the right hand equation yield a positive definite matrix as the condition of $P_k, Q_k > 0$ is satisfied in each update. For a case of extremely big state error covariance and very small process error especially for a case of stationary robot, if measurement data is missing then the statistical bound of state error covariance is shown only by the sum of previous state error covariance $P_k$ and process error $Q_k$.

Hence, the upper $\bar{P}_{k+1}$ and lower state error covariance $\underline{P}_{k+1}$ are now explicitly indicated by

$$\bar{P}_{k+1} = \underline{J}_{k+1}^{-1}$$
$$\underline{P}_{k+1} = \bar{J}_{k+1}^{-1} \quad \square$$

As shown in above *Theorem 1*, we understand that the state error covariance update is significantly being affected by the previous state error covariance and process noise covariance such that if both terms are big, then the uncertainties is increasing. This results is supported by previous results[9, 10, 11] that gives a statistical determination about the system behavior in intermittent measurement. In addition, even if measurement data is available, $P_0$ and $Q_k$ always influencing the estimation performance. As been explained before, EKF is converging $P_0$[2] which determine system efficiency. Furthermore, this fact can be obtained by analyzing (21) without $\gamma$ existence which show the normal EKF update under FIM representation. We can find that state covariance update is proportional to initial state covariance and process noise.

Further realize that, designer must consider the system design especially regarding the process noise covariance and initial state covariance to satisfy whether $P_k > Q_k$. Comparing above results with the normal EKF without intermittent measurement data lost, measurement data has an important role to give a sufficient information to the system estimation. Moreover, the FIM lower bound is now explicitly shown when there are no arrival of measurement data at $1 < k < N$.

Considering the best performance update for each observation in which the measurement data is available at all time, the state error covariance $\underline{J}_{k+1}$ should perform consistently as analyzed by S.Huang et.al[2]. In each case of stationary robot or moving robot, the state error covariance must reflect the same as analyzed by them. Besides, the convergence properties can be easily clarify to illustrate the same as normal EKF for a case of very small process noise. See (30) when the process noise $Q_k \to 0$, then the state error covariance is approximating $P_k$. The estimation indeed has achieved a desired performance level only when robot gained more information from its observations. As a matter of fact, if more measurements are made for any instant time $k$, then the FIM becomes bigger and can intensively improved the estimation. Besides, we proved that the above fact coherently guaranteeing EKF convergence as claimed by S.Huang et.al[2].

Interestingly, if the updated state error covariance shows the output which same to the lower bound, then the EKF is now very optimistic about its estimation. The RMSE result is important to evaluate whether this is acceptable or else as such condition is merely the case in real SLAM application. This results is also coherent with S.Kluge et.al[11] in which he claimed that the convergence is preserved whenever the initial state covariance and both process and measurement noises are small. If process noise and the initial state covariance is very big, from (33)-(34) it is understood that the updated state covariance becomes unintendedly big. Hence result in unbounded estimation about the states.

Different than previous literatures, it can be observed from these results that the process noise act as an important feature that significantly affect the estimation which also proved S.Kluge et.al result[11]. *Proposition 1* has explained the other variables effects to the system. However, is this characteristic remain steady even if the measurement data are lost longer? Can we guarantee the estimation to converge in a case where measurement data are not arrived for some period of time? Moreover, as each update consists of an added noises, then in a condition where measurement data are lost longer, the state error covariance can result in erroneous estimation. In other word, the uncertainties increase and substantially lead to unstable system behavior. We can summarized this effect by the following theorem.

**Theorem 2** *The updated state error covariance is increasing or decreasing proportionally to the amount of time whenever the measurement data is not available such that it is increasing by*

$$\bar{P}_{k+n} = \rho_k + n\varepsilon_k \qquad (35)$$

*where* $\varepsilon_k = (\bar{Q}_k^{-1} - \rho_k^{-1})^{-1}$ *or it is decreasing by*

$$\underline{P}_{k+n} = \rho_k - n\bar{\varepsilon}_k \qquad (36)$$

*where* $\bar{\varepsilon}_k = Q_k(\rho_k + Q_k + Q_k\rho^{-1}Q_k)^{-1}Q_k$.

**Proof** The upper bound of the updated state error covariance if a measurement data is lost at $k$ is given by

$$\begin{aligned}\bar{P}_{k+1} &= \rho_k + (\bar{Q}_k^{-1} - \rho_k^{-1})^{-1}\\ &= \rho_k + \varepsilon_k\\ \bar{P}_{k+2} &= \rho_k + \varepsilon_k + [(\bar{Q}_k^{-1} - \rho_k^{-1}]^{-1}\\ &\leq \rho_k + 2\varepsilon_k\end{aligned}$$

The updated state error covariance will increase unboundedly if there are no arrival of measurement data in longer time such that if the measurement data is not available for $n-$times, then

$$\bar{P}_{k+n} = \rho_k + n\varepsilon_k$$

Similar derivation can be obtained for the second case where the updated state error covariance yield the following equation.

$$\underline{P}_{k+n} = \rho_k - n\bar{\varepsilon}_k \quad \square$$

Hence, it can be summarized that recursive update without the existence of measurement data can contribute to the unreliable estimation especially when it is unavailable for some period of time.

## 4 Simulation Results

The above analysis are further being examine in a specified simulation case. Table 1 shows the simulation parameters which includes the process and measurement noises with an appropriate dimension. We assume that the landmarks are stationary and consists of point landmarks. We assign some points at 100[s], 500[s] and 800[s] which do not receive any measurements data for a certain time. There are 30[s] measurement data lost after 100[s], and each 1[s] and 10[s] measurement lost for each after 500[s] and 800[s] observations respectively.

Table 1: Simulation Parameters

| | |
|---|---|
| Sampling Time, T | $0.1[s]$ |
| Process noise,Q | $1 \times 10^{-6}$ |
| Observation noise, $R_{\theta_i}, R_{distance_i}$ | $R_{\theta_i} = 0.002$, $R_{distance_i} = 0.02$ |
| Robot Initial Covariance *Pvv* | $1 \times 10^{-2}$ |
| Landmarks Initial Covariance *Pmm* | 100 |

Fig.3 shows the constructed map by both normal EKF and a case of EKF with intermittent measurements. As expected, it is observable that for the case of EKF with intermittent measurement, the estimation become inconsistent whenever measurement data is not arrived. A big error is

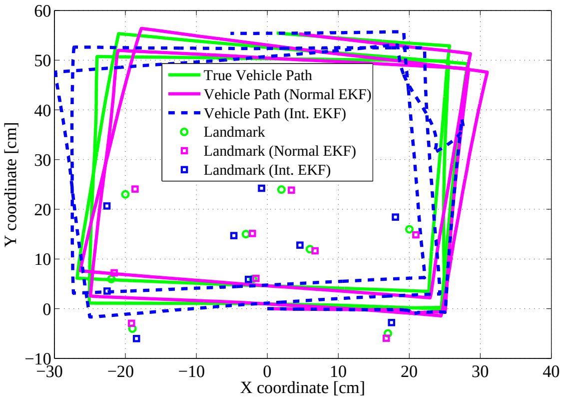


Fig. 3: Comparison between EKF-EKF with Intermittent Measurement about the constructed map

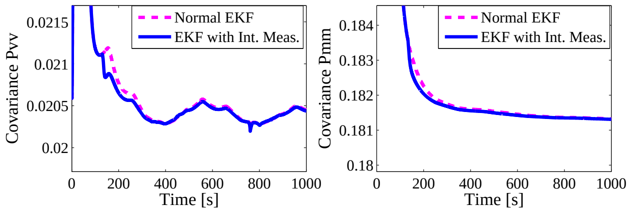


Fig. 4: Trace of the associated state error covariance between EKF-EKF with Intermittent Measurement

perceived in the respective update after no measurement data arrived at above specified time. We observed that after 100[s] where the robot lost about 30[s] regarding its measurement data, the estimation is diverging and consequently makes the robot path erroneous. This is the biggest implications compared to other specified time which lead to inconsistent estimation for both robot and landmarks estimations. In addition, the result also agree with *Proposition 1* stated in previous section.

Considering about the uncertainties, the associated state error covariances are shown in Fig.4 to demonstrate the robot and landmarks covariances. Based on probabilistic SLAM, if the state error covariance is smaller, then the estimation is improved. Surprisingly and unexpectedly, the EKF with intermittent measurement state error covariance surpassed the normal EKF state error covariance without intermittent measurement. It is noticeable that the state error covariance almost reaching but do not exceeds the lower bound as determined previously in *Theorem 1* especially about the robot state estimation. These characteristics are shown in Fig.5 for each 100[s], 500[s] and 800[s] measurement data lost. Fig.5 supports both *Theorem 1* and *Theorem 2* where we clearly understand the results when measurement data is not arrived for 1[s] and more than 1[s]. Based on Fig.5, we observed that between robot and landmarks state error covariances, robot has bigger upper and lower bounds. This is actually due to landmarks has no process noise and therefore exhibit less uncertainties than the robot has. However, this result contradict with the preceding result in Fig.3. Based on Fig.3, the robot state error covariance for EKF with intermittent measurement should be increasing and bigger than the normal EKF. We expect that this is probably due to EKF affirms that it has received a sufficient amount of information when measurement data is unavailable. This is shown by the lower bound of state error covariance as proposed by (38). Or in other words, the updated state error covariance refer to the previous state error covariance with some bounded addition of uncertainties(refer to (25)). These results also denotes explicitly that EKF become more optimistic about its estimation.

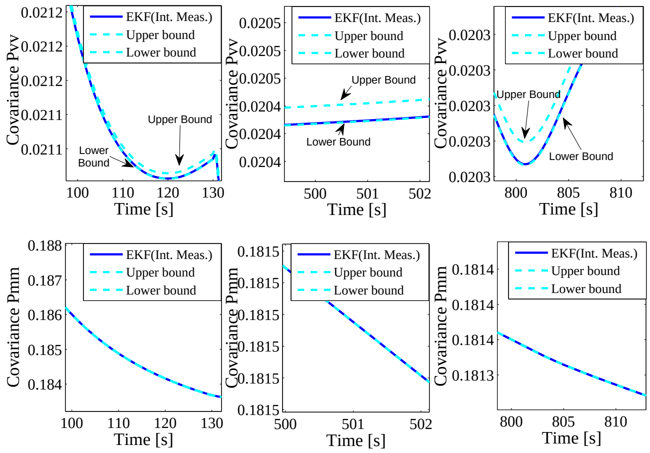


Fig. 5: Upper and lower bound of the estimation for EKF with Intermittent Measurement when measurement data is lost at 100[s], 500[s] and 800[s] for 30[s], 1[s] and 10[s]

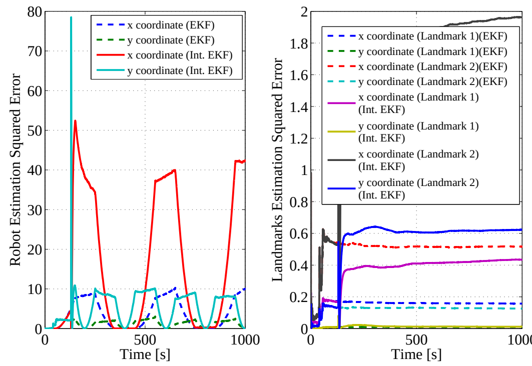


Fig. 6: Performance between EKF and EKF with Intermittent Measurement when measurement data is lost at 100[s], 500[s] and 800[s] for 30[s], 1[s] and 10[s]

Fig.6 provides a clearer descriptions about this fact. Most of the RMSE in EKF with intermittent measurement consequently become bigger since there was a *hole* in the observations data(see at 100[s], 500[s] and 800[s] whenever measurement data is missing). Despite of the results in Figs.4-5, this figure shows the true behavior of estimation. In fact, even if the initial state covariance, process and measurement noises are small, estimation can diverge and further attention is needed in such case. Observe that in this analysis, we obtain $P_{MSE} > P_{estimate}$. Consequently, this condition proves that EKF with intermittent measurement is optimistic. The updated state error covariance and the constructed map are insufficient to describe the estimation. The RMSE evaluation or any additional tools are necessitate to view the actual estimation performance.

The NEES test(Normalized Estimation Error Squared) is apply to certify the results for both cases. We include the evaluation in Fig.7. The results absolutely pictures that EKF with intermittent measurement has exhibit inconsistency. The estimation errors are growing especially when measurement data is not arrived. Hence, we conclude that, in a case of EKF with intermittent measurement, designer must carefully examine the RMSE performance to assess its performance. In an actual system and environment and due to sensors limitation, it is hard to judge each observation whether it encompasses appropriate packet information or else. Relying only to the state error covariance is insufficient as already been enclosed in above results. Fig.8 shows the effect of bigger initial state covariance, process and measurement noises. As expected, the estimation is erroneous than normal EKF.

Nevertheless, we guarantee that the EKF with intermit-

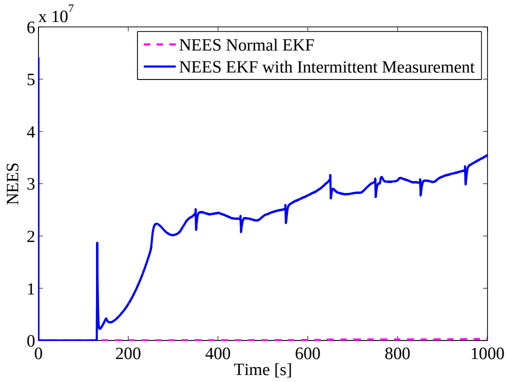


Fig. 7: NEES evaluation for both EKF and EKF with Intermittent Measurement.

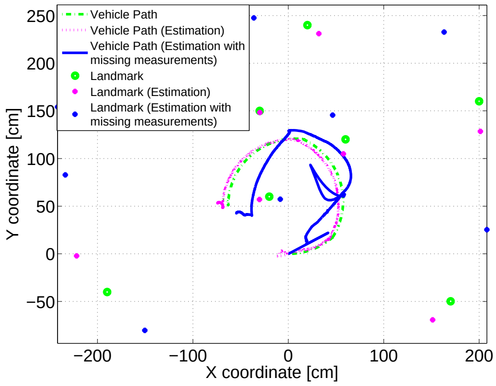


Fig. 8: Constructed map with bigger process noise, measurement noises and bigger initial state covariance

tent measurement satisfies the aforementioned upper and lower bounds despite of its optimistic behavior. The uncertainties never exceeds these bounds when measurement data is not available. Furthermore, based on the analysis and simulation results, the upper and lower bounds are determine explicitly using the FIM approach. Remark that for bigger process and measurement noise with bigger initial state covariance, the results may exhibit erroneous estimation. This condition must be considered in pursue to design a system that able to achieve a desired outcome.

## 5 Conclusion

This paper presented an analysis about EKF upper and lower bounds for the whole robot observations by utilizing the information obtained by FIM. We showed that by using FIM, it was possible to determine EKF upper and lower bounds through *Theorem 1* and *Theorem 2*. Based on these theorems, we understand that the uncertainties were increasing if measurement data was not available. Besides, the uncertainties were bounded and could be presented explicitly via FIM. Our numerical results were also supports our claims which indicated that the updated state error covariance never exceeds our proposed upper and lower bounds. There were also some certain conditions to be considered in order to affirm consistent results with our analysis. Based on the analysis, we provided a sufficient information about the updates behavior whenever a measurement data was not available. We also realized that in a case when measurement data was unavailable, even though the state error covariance was small, the estimation could show unexpected behavior. We left for the evaluation for a real application in future research development.

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