Optimal Control of Connected and Automated Vehicles at Intersections with State and Control Constraints

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Abstract—This paper addresses the optimal control problem (optimal acceleration and deceleration to minimize fuel consumption) for Connected and Automated Vehicles (CAVs) to cross an intersection with state and control constraints. First, we obtain the crossing time for the CAVs to cross the intersection with Mixed Integer Linear Programming (MILP). Then, the crossing time is used as a terminal time to formulate the optimal control problem and we utilized analytical solution (Hamiltonian analysis) to solve the problem. Finally, we simulated the proposed approach and the results show that, even with the frequently changing crossing time, the controller is capable of providing the solution to the optimal control problem.

I. INTRODUCTION

A broad utilization of vehicles on the roadway every year has caused traffic congestion to increase especially in urban areas. According to [1], primary sources of bottlenecks for the traffic congestion are intersections and merging roadways. In United States, the traffic congestion at intersections alone had forced a passenger to spend an extra 42 hours on the road yearly and wasted 3.1 billion gallon of fuel (which is caused by unnecessary acceleration or deceleration of a vehicle) in 2014 [2]. In addition, it is also reported in [3] that at least 47% of accidents in United States occur at intersections which is almost similar to the number of accidents happened in the European Union [4]. For solving these problems, many efforts have been taken to provide a better coordination and control for the movement of vehicles at the intersections. This include improving the signal phase and timing plans of the existing physical traffic lights as well as utilizing the Connected and Automated Vehicles (CAVs) technology. Because the CAVs technology can provide much faster response time compared to human-driven vehicles and capable to control their own speed precisely, therefore the aforementioned problems tend to be solved more effectively and efficiently.

There are many research efforts have been conducted to coordinate and control the movement of CAVs at intersections with the objectives to alleviate traffic congestion, reducing fuel consumption, and increasing passenger comfort where safety is becoming the hard constraint for all of these objectives. Combining all of these objectives to the problem formulation to obtain the optimal solution is very difficult and from the literature, we can see that many researchers just focusing on one of these objectives at a time but without neglecting the safety constraint. First, we view the literature with the objective to alleviate the traffic congestion and we can call the problem to achieve this objective as coordination problem. Towards solving this coordination problem, several strategies have been proposed such as reservation-based approach (heuristic) [5], minimizing the overlap of vehicles’ trajectories [6], considering the optimal time for each vehicle to arrive at the intersection [7], and job-shop scheduling based formulation [8].

Second, we view the literature with the objective to reduce fuel consumption and the problem to achieve this objective can be called as optimal control/motion planning problem. To solve this problem, usually each vehicle needs to acquire a time to cross the intersection either compute it by themselves or obtain it from a coordinator (autonomous intersection manager that replace the traffic light). For simplicity, the crossing time is usually computed based on the First-In-First-Out (FIFO) basis but the trade-off is it can cause high traffic congestion in high traffic density [9]. After obtaining the crossing time, each vehicle can solve the optimal control problem by deriving an optimal acc/deceleration to reach the obtained crossing time. In [10], a monotonic relationship between the fuel consumption and acceleration was shown where this enables us to minimize the fuel consumption by only minimizing the acceleration of each vehicle. In addition to that, [10] also have proposed an analytical method using Hamiltonian analysis to derive the optimal acc/deceleration solution. Since then, the method has been developed to include the state and control constraints [11], guaranteeing the absence of rear-end collision [11], [12], and also has been utilized in some literatures (e.g. [13], [14]).

The objective of this paper is to formulate the optimal crossing time and the optimal control for all vehicles to cross the intersection. There are some works in this direction such as [15], [16] however, the problem formulation is complex [15] and requires high computation time [16]. In this paper, we obtain the optimal crossing time with Mixed Integer Linear Programming (simple formulation) and the optimal control with Hamiltonian analysis (very fast computation). To the best of our knowledge, no results have been obtained for combining the two methods mentioned above except our previous work in [17]. In the previous work, we did not consider any state and control constraints for each vehicle and we will include the constraints in this paper. On the other hand, with Feasibility Enforcement Period (FEP), we show that the existing Hamiltonian analysis is capable of deriving the optimal acc/deceleration with the state and control constraints for each vehicle with frequently changing...
crossing time. Note that, the crossing time for existing works is fix for each vehicle (e.g., [10], [11]). However, it is necessary for the crossing time to be frequently changed to obtain more optimal solution and thus, reduce the traffic congestion significantly.

II. PROBLEM FORMULATION

A. Modeling Frameworks

![Fig. 1. Intersection model.](image)

Figure 1 shows the intersection model that we consider in this paper. It consists of four movements, $M$, i.e., southbound, $X'$, northbound, $X''$, westbound, $O'$, and eastbound, $O''$. As can be seen from the left-hand side of the figure, we group the movement of the vehicles into two phases which are Phase X, $\phi_X = \{X', X''\}$ and Phase O, $\phi_O = \{O', O''\}$. The intersection also has a Control Zone (CZ) which is depicted in a blue circle and this is the region where the CAVs can communicate with the coordinator. The length of this CZ is denoted as $L$ and is measured from the entry of the CZ, which is zero, to the entry of the Merging Zone (MZ) which is $L$. The MZ is the region where the lateral collision between vehicles from different phases ($\phi_X \neq \phi_O$) can be potentially occurred. The shape of the MZ is assumed to be a square of side $S$. Note that, the main task of the coordinator is to assign the appropriate crossing time and speed for each CAV every time a new CAV entered the CZ. As the early stage of this research, we assume that all vehicles are CAVs, and no lane changing and turning (left or right) are allowed.

When a new CAV arrives at the entry of the CZ, it sends a subscription request to the coordinator to announce its presence. Once the subscription of the newly subscribed CAV is successful, the coordinator then assigns the identification (ID), $i = 1, 2, \ldots, N(t)$ where $N(t)$ is the total number of subscribed CAVs. Note that when a new CAV is subscribed to the coordinator, all the ID of other subscribed CAVs (if exist) will be re-updated based on their distance to the MZ, i.e., the nearest CAV to the MZ will be numbered first. In some cases, there will be two or more CAVs which have the same distance at the same time (might be two or more CAVs arrive at the CZ at the same time). To deal with this problem, the coordinator can assign the ID based on the movement of each CAV, $m_i \in M$. In this paper, we prioritized the movement in an anti-clockwise direction starting from $O''$ and ending at $X''$. This means that if two or more CAVs have the same distance at the same time, then the CAV with the most priority movement will be numbered first. Note that the coordinator only computes the crossing time for all CAVs which are still subscribing to the coordinator (not yet exiting the MZ). Because of this, once a CAV left the MZ, the ID will be eliminated and the total number of subscribed CAVs will become $N(t) - 1$ (see Figure 1).

For the model of each CAV $i \in N(t)$, we only consider it as a second order linear differential equation given as

$$\ddot{p}_i = v_i(t), \quad \dot{p}_i(t_0) \text{ given}$$
$$\dot{v}_i = u_i(t), \quad v_i(t_0) \text{ given} \quad (1)$$

where $t_0 \in \mathbb{R}^+$ is the current time, $p_i(t) \in [0, L]$ is the position, $v_i(t)$ is the speed, and $u_i(t)$ is the control input (acceleration/deceleration).

For the system in (1), we need to impose some constraints on the speed and control input so that they are within a given admissible range. This can be done as follows

$$u_{i,min} \leq u_i(t) \leq u_{i,max}, \quad \text{and}$$
$$0 \leq v_{min} \leq v_i(t) \leq v_{max}, \quad \forall t \in [t_0, t_i^f], \quad (2)$$

where $u_{i,min}, u_{i,max}$ are the minimum deceleration and maximum acceleration for each CAV $i$, and $v_{min}, v_{max}$ are the minimum and maximum speed limits respectively, and $t_i^f$ is the time that CAV $i$ leaves the MZ (we also call it as final time). In this paper, we assume that all CAVs are the same and thus we can set $u_{i,min} = u_{min}$ and $u_{i,max} = u_{max}$.

Rear-end collision can potentially happen if two consecutive CAVs are traveling on the same movement (e.g., CAV 4 and CAV 5 in Figure 1). Therefore, to ensure the absence of the rear-end collision, we can define a safety distance between the two CAVs:

$$s_i(t) = p_k(t) - p_i(t) \geq \delta,$$
$$\forall t \in [t_0, t_i^f], \quad \forall m_i, m_k \in M, \quad m_i = m_k, \quad (3)$$

where $k$ is the preceding vehicle of CAV $i$ (mathematically can be described as $k = \max\{j : m_i = m_j, j = 1, \ldots, i - 1\} < i$) and $\delta$ is a predefined safety distance. From the optimal crossing time problem formulation in the next section, we will show that we can satisfy the constraint (3) at $t \in [t_i^m, t_i^f]$ only where $t_i^m$ is the time for CAV $i$ to reach the MZ. This is because satisfying the constraint for all $t \in [t_i^m, t_i^f]$ is difficult (because of the inequality constraint) and we still investigate this problem in the subject of ongoing research.

On the other hand, a lateral collision can be potentially happened inside the MZ between two CAVs traveling from different phases, for example, CAV 2 and CAV 3 in Figure 1. Before explaining the strategy on how this collision can be avoided, first, we provide the following definition:

Definition 1: For each CAV $i$, we define the set $\Gamma_i$ that includes all the time instants when a lateral collision involving CAV $i$ is possible:

$$\Gamma_i = \{t \mid t \in [t_i^m, t_i^f]\}. \quad (4)$$
From the above definition, we can see that one of the strategies to avoid the lateral collision is to prevent the two CAVs \(i, j \in N(t)\), \(i \neq j\) to be inside the MZ at the same time. This strategy can be formalized as the following constraint

\[
\Gamma_i \cap \Gamma_j = \{\}, \quad \forall \phi_i, \phi_j \in \phi, \quad \phi_i \neq \phi_j,
\]

where \(\{\}\) is the empty set.

In the following section, we will explain about obtaining the optimal crossing time for each CAV to cross the intersection. For simplicity of notation in the remainder of this paper, we will write \(p_i(t_0) = p_i^0\), \(v_i(t_0) = v_i^0\), \(v_i(t_{im}^m) = v_i^m\), \(v_i(t_{i}^f) = v_i^f\) where \(v_i(t_{im}^m)\) and \(v_i(t_{i}^f)\) are merging and final speed respectively.

III. OPTIMAL CROSSING TIME PROBLEM FORMULATION

The primary goal of obtaining the optimal crossing time is to increase the throughput of the intersection (reduce congestion). There are many strategies to obtain the crossing time for each CAV including the FIFO order based strategy. Even the FIFO strategy is very simple, however, it will only result in high traffic congestion in high traffic density. In this paper, we utilized the optimal crossing time formulation from our previous work [17]. Note that the problem is formulated based on the arrival time (we will call it as merging time hereafter), \(t_{im}^m\), of all CAVs at the MZ using Mixed Integer Linear Programming (MILP).

![Fig. 2. Merging time and final time of each CAV i.](image)

A. Feasible merging time

The model of each CAV \(i\) in (1) is imposed with constraints (2), therefore, it will not always be possible for the CAV to reach the assigned \(t_{im}^m\) by the coordinator. To avoid such a scenario, the following constraint is imposed on \(t_{im}^m\):

\[
t_{im}^m \geq t_{im,min},
\]

where \(t_{im,min}\) is the fastest time (lower bound of \(t_{im}^m\)) for each CAV to reach the MZ and from [7], it can be computed as follows:

\[
t_{i}^1 = \min\left\{t_{max} - t_{0}^0, \frac{v_{i}^m - t_{0}^0}{u_{max}}\right\},
\]

\[
t_{i}^2 = \max\left\{\frac{d_i}{v_{max}} - \frac{(v_{max} - v_0)^2}{2u_{max}v_{max}}, 0\right\},
\]

\[
t_{im}^m = t_0 + t_{i}^1 + t_{i}^2.
\]

where \(d_i = L - p_i^0\) is the distance of each CAV \(i\) to the MZ at current time \(t_0\), and \(v_{imax} = \sqrt{(v_0^m)^2 + 2u_{max}d_i}\) is the maximum merging speed that the CAV can reach. Note that (7) is derived from the equation of motion where \(t_{i}^1\) is to determine either the CAV can reach the \(t_{im}^m\) with the maximum speed or below and \(t_{i}^2\) is to determine the cruising time with the maximum speed (if the CAV reaches the maximum speed before reaching \(t_{im}^m\) or else it is zero).

The slowest merging time can be introduced for each CAV \(i\) to reach the merging zone to avoid the CAV from getting too slow. To avoid such a scenario, the following constraint is imposed on \(t_{im}^m\):

\[
t_{im}^m \leq t_{im,max},
\]

where \(t_{im,max}\) is the slowest time (upper bound of \(t_{im}^m\)) for each CAV to reach the MZ. In this paper, we assume \(t_{im,max}\) to be arbitrarily maximum allowed time.

Please note that in this paper, we assume that all the merging speeds, \(v_{im}^m\) recommended by the coordinator to the CAVs are equal to \(v_{max}\) and thus, we can obtain the final time as \(t_{i}^f = t_{im}^m + S/v_{max}\). This assumption is possible if the length of CZ, \(L\) is large enough.

B. Avoiding collisions at the merging time

As mentioned previously, we have two collision avoidance that need to be addressed. First is avoiding the rear-end collision and the second one is avoiding the lateral collision. In our previous work [17], we assumed that the speed of all CAVs inside the MZ is constant and thus we can guarantee the absence of rear-end collision at the merging time by setting \(t_{im}^m - t_{ik}^m \geq \delta/v_{ik}^m\) where \(v_{ik}^m\) is the merging speed of the preceding vehicle of CAV \(i\). Since all \(v_{ik}^m\) are assumed to be \(v_{max}\), a similar strategy is utilized, namely

\[
t_{im}^m - t_{ik}^m \geq h_R, \quad \forall m_i, m_k \in M, \quad m_i = m_k,
\]

where \(h_R = \delta/v_{max}\) is the headway time to avoid rear-end the collision between CAV \(i\) and CAV \(k\).

As stated in (5), the lateral collision can be avoided by allowing the CAVs traveling from different movements to...
cross the intersection one at a time. For this case, we can avoid the collision as follows
\[ t^m_i - t^m_j \geq h_L \]
OR
\[ t^m_i - t^m_j \geq h_L, \]
\[ \forall \phi_i, \phi_j \in \phi, \phi_i \neq \phi_j, \]  
(10)
where \( h_L = S/v_{max} \) is the headway time to avoid the lateral collision between CAV \( i \) and CAV \( j \).

Note that the OR logic in (10) enables the coordinator to determine the optimal sequence for the CAVs to cross the intersection, i.e., either the CAV \( i \) or CAV \( j \) crosses first. In MILP formulation, the OR logic cannot be utilized. However, we can convert the inequality constraint in (10) to AND combination using the big-M method as follows
\[ t^m_i - t^m_j + M_{big} B_i \geq h_L \]
AND
\[ t^m_j - t^m_i + M_{big} (1 - B_i) \geq h_L, \]
\[ \forall \phi_i, \phi_j \in \phi, \phi_i \neq \phi_j, \]  
(11)
where \( M_{big} \) is a very big constant number (see [7] to appropriately choose this number) and \( B_i, i = 1, 2, ..., \) no. of constraints is binary variables.

C. Objective function

The objective of our MILP formulation is to minimize the total time for all CAVs to reach the MZ. By minimizing this objective, we can ensure that all CAVs can reach the merging time as fast as possible which is technically can reduce the traffic congestion. Formally, the objective can be defined as:
\[ \min \sum_{i=1}^{N(t)} t^m_i, \]
subject to: (6), (9), (11).  
(12)

The MILP problem in (12) can be solved using various kind of available MILP solvers such as intlinprog function from Matlab or CPLEX from IBM (provide much faster solution). In this research, we started to use the intlinprog function first and will use the CPLEX in the next research. Note that, we denote the solution of (12) as \( t^m_{i,N(t)} \) where this solution is used to specify the terminal time of the energy minimization problem in the next section.

IV. ENERGY MINIMIZATION PROBLEM FORMULATION

A. Objective function

In [10], a monotonic relationship between the fuel consumption and acceleration was shown. This relationship enables us to formulate the energy minimization problem (in this case, is the fuel consumption) by only minimizing the acceleration of each CAV. Therefore, the objective function of the energy minimization problem can be defined as
\[ \min u_i \int_{t_0}^{t^m_i} u^2(t) dt, \]
subject to: (1), (2), \( p_i(t^m_i) = L_i \),  
(13)

After receiving the \( t^m_i \) and \( v^m_i \) from the coordinator, (13) can be solved by onboard computer of each CAV \( i \) to obtain the optimal acceleration/deceleration to reach the \( t^m_i \) with minimum energy consumption. There are two options to solve the problem which is by utilizing the analytical solution [11] or numerical solution [16] (by discretizing (13) and the system in (1)). Because the analytical solution can promise much faster computation time (which is one of the demands in the CAV technology), this motivates us to utilize the analytical solution.

B. Analytical solution

In this paper, we utilized the analytical solution (Hamiltonian analysis) from [11] to solve (13) which is also a standard methodology used in optimal control problems. To avoid the infeasibility in the solution, the following assumption should be noted.

Assumption 2: When the crossing time \( t^m_i \) is re-solved, none of the constraints is active.

Note that, we introduce a Feasibility Enforcement Period (FEP) where in this period, the CAV will slow down or speed up so that the assumption 2 is not being violated.

From (13), (1), and (2), for each CAV \( i \) the Hamiltonian function with the state and control constraints is given as follows
\[ H_i(t, p(t), v(t), u(t)) = \frac{1}{2} u^2 + c^p_i \cdot v_i + c^v_i \cdot u_i \]
\[ + \lambda^2_i \cdot (u_i - u_{max}) + \lambda^3_i \cdot (u_{min} - u_i) \]
\[ + \lambda^4_i \cdot (v_i - v_{max}) + \lambda^5_i \cdot (v_{min} - v_i), \]  
(14)
where \( c^p_i \) and \( c^v_i \) are the co-states, and \( \lambda^T \) is a vector of Lagrange multipliers with
\[ \lambda^1_i = \begin{cases} > 0, & u_i(t) - u_{max} = 0, \\ = 0, & u_i(t) - u_{max} < 0, \end{cases} \]
(15)
\[ \lambda^2_i = \begin{cases} > 0, & u_{min} - u_i(t) = 0, \\ = 0, & u_{min} - u_i(t) < 0, \end{cases} \]
(16)
\[ \lambda^3_i = \begin{cases} > 0, & v_i(t) - v_{max} = 0, \\ = 0, & v_i(t) - v_{max} < 0, \end{cases} \]
(17)
\[ \lambda^4_i = \begin{cases} > 0, & v_{min} - v_i(t) = 0, \\ = 0, & v_{min} - v_i(t) < 0. \end{cases} \]
(18)

The Euler-Lagrange equations become
\[ \dot{c}^p_i = - \frac{\partial H_i}{\partial p_i} = 0, \]  
(19)
\[ \dot{c}_i^p = -\frac{\partial H_i}{\partial v_i} = \begin{cases} -c_i^p, & v_i(t) - v_{\text{max}} < 0 \\ -c_i^p - \lambda_i^3, & v_i(t) - v_{\text{max}} = 0 \\ -c_i^p + \lambda_i^4, & v_i(t) - v_{\text{max}} = 0. \end{cases} \] (20)

The condition for optimality is
\[ \frac{\partial H_i}{\partial u_i} = u_i + c_i^p + \lambda_i^1 + \lambda_i^2 = 0. \] (21)

For ease of explanation, we address the problem (14)-(20) in step by step approach.

STEP 1: Control and state constraints are not violated
For every \( t > t_0 \), the computation will always start by finding the optimal control and state where the constraints are not violated. In this case, \( \lambda_i^1 = \lambda_i^2 = \lambda_i^3 = \lambda_i^4 = 0 \). From (21), the optimal control is obtained as
\[ u_i + c_i^p = 0. \] (22)

and the Euler-Lagrange equations yield (19) and
\[ c_i^p = -\frac{\partial H_i}{\partial v_i} = -c_i^p. \] (23)

Solving \( c_i^p \) in (22) (can be obtained by solving (19) and the first condition in (20)) yield the optimal input
\[ u_i^*(t) = a_i t + b_i, \] (24)

where \( a_i \) and \( b_i \) are integration constant. Substituting (24) into system (1) gives us the optimal speed and position
\[ v_i^*(t) = \frac{1}{2} a_i t^2 + b_i t + c_i, \] (25)
\[ p_i^*(t) = \frac{1}{6} a_i t^3 + \frac{1}{2} b_i t^2 + c_i t + d_i, \] (26)

where \( c_i \) and \( d_i \) are integration constants. By using the initial and final conditions of the problem (13), i.e., \( p_i^0, v_i^0, p_i^m = L \) and \( v_i^m = v_{\text{max}} \), we can solve the constants above by the following matrix (in the form of \( T_i b_i = q_i \)):
\[ \begin{bmatrix} \frac{1}{2} (t_0)^3 & \frac{1}{2} (t_0)^2 & t_0 & 1 \\ \frac{1}{2} (t_1)^3 & \frac{1}{2} (t_1)^2 & t_1 & 1 \\ \frac{1}{2} (t_m)^3 & \frac{1}{2} (t_m)^2 & t_m & 1 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} = \begin{bmatrix} p_i(t_0) \\ v_i(t_0) \\ p_i(t_m) \\ v_i(t_m) \end{bmatrix}, \] (27)

where \( t_m \) is obtained by solving problem (12). Since the above matrix is solved (by rearranging the above matrix to \( b_i = T_i^{-1} q_i \)) for every \( t > t_0 \), we can rewrite the four constants as \( a_i(t, p_i(t), v_i(t)), b_i(t, p_i(t), v_i(t)), c_i(t, p_i(t), v_i(t)), d_i(t, p_i(t), v_i(t)) \).

STEP 2: Control Constraint is violated
Suppose that the obtained optimal input in (24) is higher than \( v_{\text{max}} \) at \( t = t_1 \) (while \( v_{\text{min}} \leq v_i(t) \leq v_{\text{max}} \)), then (24) becomes
\[ u_i^*(t) = v_{\text{max}}, \forall t \geq t_1, \] (28)

and by substituting (28) into the system (1), we can get the optimal speed and position, namely
\[ v_i^*(t) = u_{\text{max}} t + f_i, \] (29)
\[ p_i^*(t) = \frac{1}{2} u_{\text{max}} t^2 + f_i t + e_i, \forall t \geq t_1, \] (30)

where \( f_i \) and \( e_i \) are the integration constants and can be computed from the obtained speed and position in (25) and (26) respectively.

STEP 3: Control and State Constraints are violated
Suppose that the obtained optimal speed in (29) is higher than \( v_{\text{max}} \) at \( t = t_2 > t_1 \) (while \( u_i(t) = u_{\text{max}} \)), then (29) and (28) become \( v_i^*(t) = v_{\text{max}} \) and \( u_i^*(t) = 0 \) respectively. For the optimal position, it becomes
\[ p_i^*(t) = v_{\text{max}} t + r_i, \forall t \geq t_2 \] (31)

where \( r_i \) is the integration constant and can be computed from the obtained position (30). Note that, for the case where the state constraint is violated, i.e., the speed higher than \( v_{\text{max}} \) (while \( v_{\text{min}} \leq u_i(t) \leq u_{\text{max}} \)), the optimal input, speed and position also become \( u_i^*(t) = 0, v_i^*(t) = v_{\text{max}} \) and (31) respectively.

Note that, if \( u_i^*(t) = 0 \), there is a possibility that the state constraint becomes inactive again, i.e., \( v_{\text{min}} \leq v_i(t) \leq v_{\text{max}} \). For this problem, see [11].

STEP 4: For the minimum constraint cases
For the case where the input and state violate the minimum constraints, the same steps from step 2 and step 3 can be repeated.

V. SIMULATION RESULTS
In this section, the proposed solution is simulated using the crossing scenario presented previously in Matlab. In the simulation, we set the length of CZ, \( L = 300m \) and size of MZ, \( S = 30m \). The size of MZ might seems to be large but considering the multi-lane intersections in the city, the size is realistic. To ensure the absence of collisions between the CAVs in the same movement during the merging time, the safety distance, \( \delta \) is set to be 10m. For the maximum and minimum speed limits, they are set to be \( 13m/s \) and \( 0m/s \) respectively. On the other hand, \( 2m/s^2 \) and \( -2m/s^2 \) are set to be the maximum and minimum acceleration (control input) respectively. \( M_{\text{big}} \) needs to be sufficiently large and by following the provided guide, we chose \( M_{\text{big}} = 2000 \). Finally, the arrival of CAVs at the CZ is modeled using the Poisson distribution (1000 vehicles/hour) and the arrival speed of all CAVs at the CZ are all set to be 11.1m/s.

Figure 3, 4, and 5 show the control input, speed, and position of first 10 CAVs arrived at the CZ. As can be seen from Figure 3 and Figure 4, all the CAVs are traveling within the given speed and acceleration limits. In Figure 5, the arrival time between the CAVs at the CZ is very short (since we set the arrival rate to be 1000 vehicles/hour). This is because, if we set the arrival rate to be small, we cannot see the CAVs rearrange themselves as in Figure 5 (e.g., CAV 2, CAV 4, CAV 5, CAV 6). This means that in high traffic.
density, the CAVs will try to rearrange themselves so that the merging time can be minimized and thus reduce the traffic congestion.

![Control input of 10 CAVs.](image1)

![Speed of 10 CAVs.](image2)

![Position of 10 CAVs.](image3)

VI. CONCLUSIONS

This paper has addressed the optimal control problem for CAVs to cross an intersection with state and control constraints. To obtain the optimal control (acc/deceleration) for the CAVs to cross the intersection, first we obtain the crossing time for each CAV and formulated the crossing time with MILP. The crossing time is then used as a terminal time for the optimal control problem. To solve the optimal control problem, we utilized the Hamiltonian analysis. The results showed that, even with the frequently changing crossing time, the controller was capable to handle the problem. For future research, we are going to address the rear-end collision avoidance from the time that the CAV reaches the CZ until it leaves the MZ. Besides that, we also will compare the proposed method with the existing one with several rates of vehicle arrivals to validate the performance (in terms of average travel time and fuel consumption) of the controller in the future research.

REFERENCES