Simultaneous Localization and Mapping Problem
via the $H_{\infty}$ Filter with a Known Landmark

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Abstract: This paper deals with the simultaneous localization and mapping (SLAM) problem via the $H_{\infty}$ filter with a known landmark. By adding the observation of a known landmark to those of unknown landmarks, the linearized SLAM model satisfies its observability, and its estimation accuracy is improved. To prove the improvement theoretically, this paper shows that the determinant of the estimated error covariance matrix with the observation of a known landmark becomes small compared with that of the conventional $H_{\infty}$ filter. The convergence of the error covariance matrix is also proven in this paper. With simulations and experimental results, we confirm that the derived theorems for the convergence are correct and that we can accurately estimate the state of the robot and the environment.

Keywords: SLAM, $H_{\infty}$ filter, Observability

1. INTRODUCTION

Recently, the number of autonomous mobile robots has been increasing. Beside these robots help and reduce our work, they achieve hard tasks such as exploring disaster sites. In order to carry out these tasks autonomously, a mobile robot has to estimate its own position and construct a map around it simultaneously, a problem referred to as the “simultaneous localization and mapping (SLAM) problem” [1].

For the SLAM problem, the probabilistic approach has been studied widely. Among various kinds of the probabilistic approach to the SLAM problem, the method with the extended Kalman Filter (EKF) is one of the earliest methods. Because of the facility of its use, the EKF is still commonly used, and its convergence and consistency has been shown in [2]. However, in using the EKF, we have to set some assumptions for its accurate estimation. Then, we use the $H_{\infty}$ filter. The $H_{\infty}$ filter approach to the SLAM problem has been proposed in [3, 4]. These papers show the validity of the $H_{\infty}$ filter-based SLAM and that the estimation accuracy is improved under the non-Gaussian noise condition. Moreover, in the paper [5], we proposed the method to avoid finite escape time which is the divergence of the error covariance matrix in the $H_{\infty}$ filtering problem.

The observability is another cause which makes the SLAM problem difficult. The usual SLAM system which consists of a mobile robot and unknown landmarks is only partially observable. To satisfy its full observability, the SLAM problem with known landmarks has been proposed in [6-8]. These paper also shows that its estimation accuracy is improved via the EKF with the observation of the known landmarks. However, the theoretical proof of its improvement is not shown in these papers.

Hence, we propose the solution to the SLAM problem via the $H_{\infty}$ filter with a known landmark. As proposed in [6-8], we consider the SLAM problem which consists of unknown landmarks and one landmark that position is known a priori. In this paper, to offer the theoretical proof of the reduction of uncertainties with the observation of the known landmark, we show the error covariance matrix with a known landmark observation becomes smaller than that composed of only unknown landmarks observations. As a result of diminishing the error covariance matrix, our proposed method can be used with the smaller $\gamma$, and thus we can expect that the estimation accuracy is further improved. We also show the convergence of the estimated error covariance matrix as the algorithm is updated when the stationary robot continues its observations. Finally, we represents the validity and usefulness of our proposed method with simulation and experimental results.

2. PROBLEM FORMULATION

The system configuration and the general model of the SLAM problem are shown in Figs. 1 and 2, respectively. Beside the usual system of the SLAM problem which consists of a mobile robot and $M$ unknown landmarks, our proposed system has one known landmark. We consider the problem in which the robot uses observation data that contains uncertainties to simultaneously estimate its own position and that of the $M$ unknown landmarks in the $X$-$Y$ plane. The SLAM problem is represented by a process model and an observation model.

The process model explains the state transition of the robot and landmarks. We define the state of the system $x_k$ as follows.
\( x_k := \begin{bmatrix} \theta_{R_k} & x_{R_k} & y_{R_k} & p_{all}^T \end{bmatrix}^T \in \mathbb{R}^{3+2M} \)  \( l \) \hfill (1)

\[ p_{all} := \begin{bmatrix} x_1 & y_1 & \cdots & x_M & y_M \end{bmatrix}^T \in \mathbb{R}^{2M} \] \hfill (2)

where \( \theta_{R_k}, x_{R_k} \) and \( y_{R_k} \) are the attitude angle and \( X-Y \) position of the robot, and \( x_i \) and \( y_i \) \( (i = 1, 2, \cdots, M) \) are the \( X-Y \) position of the \( i \)th landmark, respectively. Since the known landmark need not be estimated, the state is composed of a robot and only the unknown landmarks. With a straight-line approximation using Euler’s method, the equation for updating the state is represented as follows.

\[ x_{k+1} = f(x_k, v_k, \omega_k) + e_{1k} \] \hfill (3)

\[ f(x_k, v_k, \omega_k) := \begin{bmatrix} \theta_{R_k} + T \omega_k x_{R_k} + Ty_k \cos \theta_{R_k} & 0 \\ y_{R_k} + Ty_k \sin \theta_{R_k} \end{bmatrix} \] \hfill (4)

where \( f \in \mathbb{R}^{3+2M} \) is a nonlinear function that defines the state transition, in which \( v_k, \omega_k \) and \( T \) are the robot velocity input, turning rate input and the sampling time, respectively. Meanwhile, landmarks are assumed to be stationary and there is no state transition of their positions. In Eq. (3), \( e_{1k} \in \mathbb{R}^{3+2M} \) is the process noise whose mean and covariance are \( 0 \) and \( Q_1 \).

The observation model represents the relative position between the robot and each of the landmarks. The observation equations of the \( i \)th landmark are given as follows.

\[ y_{k} = h_i(x_k) + e_{2ik} \] \hfill (5)

\[ h_i(x_k) = \begin{bmatrix} q_i \\ r_i \end{bmatrix} = \arctan \left( \frac{y_i - y_{R_i}}{x_i - x_{R_i}} \right) \] \hfill (6)

where \( h_i(x_k) \in \mathbb{R}^{2 \times 1} \) is the observation function of the \( i \)th landmark which consists of the relative distance, \( r_i \) and relative angle, \( q_i \) between the robot and the \( i \)th landmark, and \( e_{2ik} \in \mathbb{R}^{2 \times 1} \) is its observation noise with zero mean and covariance \( R_i \). In the same way, \( y_{p_i} \) is defined as the observation output of the known landmark.

The summarized observation output of the \( M \) unknown landmarks and the one known landmark is as follows.

\[ y_k = \begin{bmatrix} y_{1k}^T & \cdots & y_{Mk}^T & y_{p_k}^T \end{bmatrix}^T = h(x_k) + e_{2p}^T \] \hfill (7)

where \( h(x_k) \in \mathbb{R}^{2(M+1) \times 1} \) is the nonlinear observation function and \( e_{2p} \in \mathbb{R}^{2(M+1) \times 1} \) is the summarized observation noise with zero mean and covariance \( R_p \).

Here, not to consider the problem of data association, the following assumption is introduced.

Assumption 1: The robot can simultaneously observe and identify each and every landmark.

### 3. \( H_\infty \) Filter-Based SLAM

#### 3.1 \( H_\infty \) Filtering Problem and Finite Escape Time

In terms of a linear state vectors \( x_k \), the finite-time \( H_\infty \) filtering problem seeks to estimate \( \hat{x}_k = \hat{x}_k^i \) \( (k = 0, 1, \cdots, N) \), which satisfies the following conditional equation for \( \gamma > 0 \). Note that we use a different typeface such as \( x_k \) to distinguish the symbols used in the following equation from the others.

\[
\sup_{\mathbf{x}_0 \in \mathbb{R}^{3+2M}} \sum_{k=0}^{N} \frac{||\mathbf{x}_k - \mathbf{\hat{x}}_k||^2}{\gamma^2} < \infty
\]

where \( P_0 > 0, Q_k > 0 \) and \( R_k > 0 \) are the weighting matrices for the initial state \( x_0 \) and the bounded energy noises \( w_k \) and \( v_k \), respectively. Equation (8) represents that the ratio of the energy of the estimated error and those of the noises is smaller than a certain value \( \gamma^2 \) for any bounded energy noise. This equation shows that, when using the \( H_\infty \) filter, the smaller the design parameter \( \gamma \), the better the estimation accuracy. However, if \( \gamma \) is too small, a unique solution to the \( H_\infty \) filtering problem does not exist, and eventually, finite escape time, the divergence of the error covariance matrix occurs [4, 5].

#### 3.2 The Proposed \( H_\infty \) Filter Algorithm in SLAM

We assume the following to employ the \( H_\infty \) filter.

Assumption 2: The process noise \( e_{1k} \) and the observation noise \( e_{2k} \) are independent of each other. Moreover, they are bounded deterministic noises, whose accumulated energies for a given \( N \) satisfy the following.

\[
\sum_{k=0}^{N} ||e_{1k}||^2 < \infty, \quad \sum_{k=0}^{N} ||e_{2k}||^2 < \infty
\]

In a similar way with the EKF [2], we use the Jacobian matrices of the state transition function and observation function. Each matrix is defined as follows, respectively.

\[
F_k := \frac{\partial f(x, v, \omega)}{\partial x} = \begin{bmatrix} F_R & 0_{3 \times 2M} \\ 0_{2M \times 3} & I_{2M \times 2M} \end{bmatrix}
\]

\[
H_k := \frac{\partial h(x)}{\partial x} = \begin{bmatrix} \begin{bmatrix} \mathbf{A}_p \end{bmatrix} & \mathbf{H}_p \end{bmatrix}^T
\]

where

\[
F_R := \begin{bmatrix} 1 & 0 & 0 \\ -v_k T \sin \theta_{R_k} & 1 & 0 \\ v_k T \cos \theta_{R_k} & 0 & 1 \end{bmatrix}
\]

\[
H_k := \begin{bmatrix} e^{-T} & -e^{-T} & \cdots & -e^{-T} \\ -A_1^T & -A_2^T & \cdots & -A_M^T \end{bmatrix}
\]

\[
\mathbf{e} := \begin{bmatrix} 1 \\ \cdots \\ 0 \end{bmatrix}, \quad \mathbf{A}_i := \begin{bmatrix} -d_{x_i} & -d_{y_i} \\ d_{x_i} & d_{y_i} \end{bmatrix} \ (i = 1, 2, \cdots, M, p)(15)
\]

where \( \theta_{R_k} \) is the attitude angle of the robot, and \( d_{x_i} := x_i - x_{R_i}, \ d_{y_i} := y_i - y_{R_i} \), and \( r_i > 0 \) defined in Eq. (6) are the relative distances between a robot and each of the landmarks. In the SLAM problem, these values are estimated via the filter described below.

To obtain a successful estimation via the \( H_\infty \) filter, its observability must be satisfied. However, as described in [6-8], a system composed of observations of only unknown landmarks is only partially observable. If we add a known landmark to the observation matrix, we obtain the following lemma for the observability.

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Lemma 1: If Assumption 1 holds, the linearized SLAM model with a known landmark is observable, i.e., the rank of the observability matrix $G_O$ is $2M + 3$, which is the dimension of the state $x_k$.

$$G_O := \left[ H_k^T (H_k^T F_k^2)^T \cdots (H_k^T F_k^{2M+2})^T \right]^T \tag{16}$$

**Proof:** Since this can be proved in the same way as that of Conjecture 4 in [7], the details are omitted.

Here, the necessary and sufficient condition for the existence of a unique solution to the proposed $H_o$ filter is to satisfy the following two conditions.

1. The following Riccati equation has a positive definite solution:
   $$P_{k+1} = F_k P_k \Psi_k^{-1} F_k^T + Q_k > 0 \tag{17}$$

   $$\Psi_k := I + (H_k^T R_k^{-1} H_k - \gamma^{-2} I) P_k \tag{18}$$

2. The following equation has a positive definite solution:
   $$P_k^{-1} + H_k^T R_k^{-1} H_k - \gamma^{-2} I > 0, \quad k = 0, 1, \cdots, N \tag{19}$$

If the above two conditions are satisfied, the $H_o$ filter algorithm with the observation of a known landmark for estimating the states of the robot and the unknown landmarks is given in the following three recursive steps.

Here, $\tilde{x}_{k+1|k}$ and $\hat{x}_{k+1|k+1}$ represent a priori and a posteriori estimated values, respectively.

**Step 1:** Prediction
$$\tilde{x}_{k+1|k} = f(\hat{x}_{k|k}, v_k, a_k), \quad \hat{y}_{k+1|k} = h^T(\tilde{x}_{k+1|k}) \tag{20}$$

**Step 2:** Observation
$$\tilde{y}_{k+1} = y_{k+1} - \hat{y}_{k+1|k} \tag{21}$$

**Step 3:** Update
$$\hat{x}_{k+1|k+1} = \tilde{x}_{k+1|k} + K_{k+1} \tilde{y}_{k+1} \tag{22}$$

The filter gain $K_{k+1}$ and the estimated state error covariance matrix $P_k$ are given as follows.

$$K_k = P_k H_k^T (H_k^T P_k H_k^T + R_k)^{-1} \tag{23}$$

$$P_{k+1} = F_k P_k \Psi_k^{-1} F_k^T + Q_k \tag{24}$$

$$\Psi_k = I + (H_k^T R_k^{-1} H_k - \gamma^{-2} I) P_k \tag{25}$$

### 4. CONVERGENCE PROPERTY

In the probabilistic approach to the SLAM problem, the convergence of the state error covariance matrix $P_k$ is the parameter of the confidence of the estimation, and its determinant $|P_k|$ is a measure of the uncertainty [1, 8].

#### 4.1 Reduction of uncertainties

As we can expect and described in [6], the estimation accuracy is improved when a known landmark is included in the observations. In order to prove it theoretically, in this subsection, we show that in this case the determinant of the state error covariance matrix becomes small.

To compare the error covariance matrix, $P_k$ is defined as the error covariance matrix for the conventional $H_o$ filter (HF) and consists only of unknown landmarks. Then, the following lemma is obtained for the error covariance matrix with the observation of a known landmark.

Lemma 2: Suppose that Assumptions 1 and 2 hold and that each unique solution of the HF and the proposed $H_o$ filter which includes the observation of a known landmark exist. Then, assuming that the initial error covariance matrices for the both filter are the same, $P_0 = P_0^*$, the following inequality regarding $P_k$ and $\hat{P}_k$ holds.

$$P_k \leq \hat{P}_k, \quad k = 1, 2, \cdots \tag{26}$$

**Proof:** From Eqs. (24) and (25), the error covariance matrix for the proposed $H_o$ filter is updated as follows.

$$P_{k+1} = F_k P_k (I + (H_k^T R_k^{-1} H_k - \gamma^{-2} I) P_k)^{-1} F_k^T + Q_k \tag{27}$$

$$\hat{P}_{k+1} := (P_k^{-1} + H_k^T R_k^{-1} H_k - \gamma^{-2} I)^{-1} F_k + Q_k \tag{28}$$

Similarly, the error covariance matrix for the HF is updated as follows.

$$P_{k+1} = F_k P_k (I + (H_k^T R_k^{-1} H_k - \gamma^{-2} I) P_k)^{-1} F_k^T + Q_k \tag{29}$$

$$\hat{P}_{k+1} := (P_k^{-1} + H_k^T R_k^{-1} H_k - \gamma^{-2} I)^{-1} F_k + Q_k \tag{30}$$

In the above equations, $\hat{H}_k$ is the Jacobian matrix of the observation function consisting of the only unknown landmarks and given as follows.

$$H_k := \begin{bmatrix} H_e & H_p \end{bmatrix} \tag{31}$$

$$H_e := \begin{bmatrix} -e^T & -e^T & \cdots & -e^T \end{bmatrix} \tag{32}$$

$$H_p := \text{block diag} \{A_1, A_2, \cdots, A_M \} \tag{33}$$

Comparing Eqs. (27) and (29), the difference between the error covariance matrices is represented as follows.

$$P_{k+1} - \hat{P}_{k+1} = F_k P_k (I + (H_k R_k^{-1} H_k - \gamma^{-2} I) P_k)^{-1} F_k$$

$$\hat{P}_{k+1} := (P_k^{-1} + H_k R_k^{-1} H_k - \gamma^{-2} I)^{-1} F_k + Q_k \tag{34}$$

Here, from Eq. (34) and (29), we obtain the following.

$$\hat{P}_k^{-1} - \hat{P}_k = P_k^{-1} - P_k^{-1} - D_k \tag{35}$$

$$D_k := H_k^T R_k^{-1} H_k - H_k^T R_k^{-1} H_k \tag{36}$$

Moreover, from Eqs. (11) to (14) and (31) to (33), $D_k$ becomes the following.

$$D_k = H_k^T R_k^{-1} H_k - H_k^T R_k^{-1} H_k = \begin{bmatrix} e^T R_k^{-1} e & e^T R_k^{-1} A_p & 0_{1 \times 2M} \\ A_p^T R_k^{-1} e & A_p^T R_k^{-1} A_p & 0_{2M \times 2M} \end{bmatrix} \geq 0 \tag{37}$$

Here, $A_p$ defined in Eq. (15) is calculated with the state of the known landmark and its observed value.

Then, using induction, we prove that Eq. (26) holds for all steps $k (k = 1, 2, \cdots)$. First, consider the case of $k = 0$. Since the initial error covariance matrices are the same, $P_0 = P_0$, Eq. (35) becomes the following.

$$P_0^{-1} - \hat{P}_0^{-1} = P_0^{-1} - P_0^{-1} + D_0 = D_0 \geq 0 \tag{38}$$

Since $P_0$ is a positive definite matrix from Eq. (19) and so $\hat{P}_0$ is also, $P_0 - \hat{P}_0 \leq 0$ holds from the above equation. Therefore, Eq. (34) with $k = 1$ becomes the following.

$$P_1 - \hat{P}_1 = F_k (P_0 - \hat{P}_0) F_k^T \leq 0 \tag{39}$$

We prove that Eq. (26) holds for the case of $k = 1$.

$$P_1 \leq \hat{P}_1 \tag{40}$$
Next, suppose that Eq. (26) holds for the case of \( k = j \).

\[ P_j \leq \tilde{P}_j \quad (41) \]

Since \( P_j \) and \( \tilde{P}_j \) are positive definite matrices, \( P_j^{-1} - \tilde{P}_j^{-1} \geq 0 \) holds. In addition, from Eq. (37), \( D_j \) is also a semi-positive definite matrix. Hence, for the case of \( k = j \), Eq. (35) becomes the following.

\[ F_j^{-1} - \tilde{F}_j^{-1} = P_j^{-1} - \tilde{P}_j^{-1} + D_j \geq 0 \quad (42) \]

Thus the equation \( \hat{F}_j - \tilde{F}_j \leq 0 \) holds, as in the case of \( k = 0 \). Therefore, the following equation is obtained.

\[ P_{j+1} - \tilde{P}_{j+1} = \hat{F}_j(P_j - \tilde{P}_j) F_j \leq 0 \quad (43) \]

Hence, we have also proved that Eq. (26) holds for the time step \( j + 1 \).

\[ P_{j+1} \leq \tilde{P}_{j+1} \quad (44) \]

From the above, we have inductively proved that Eq. (26) holds for all time steps \( k \) \((k = 1, 2, \cdots)\).

From Lemma 2, we have that \( P_k \leq \tilde{P}_k \) hold for all time steps \( k \) \((k = 1, 2, \cdots)\) as long as the solutions of both filters exist. Then, using a property of the determinants of positive definite matrices, the following inequality regarding their determinants \( |P_k| \) and \( |\tilde{P}_k| \) also holds.

\[ |P_k| \leq |\tilde{P}_k|, \quad k = 1, 2, \cdots \quad (45) \]

In addition, \( D_k \) in Eq. (36) is also a semi-positive definite matrix when there are multiple known landmarks. Therefore, Lemma 2 can be extended for such a case.

### 4.2 Convergence of the error covariance matrix

From Lemma 2 and Eq. (45), the determinant of the error covariance matrix becomes small with the observations of a known landmark. However, when a robot moves, the Jacobian matrices \( F_k \) and \( H_k^T \) change at each time step, and there is always process noise. Furthermore, since these Jacobian matrices are calculated by using estimated values, they change minutely even if a robot is stationary. Then, in order to confirm the convergence of our proposed system, we assume for a stationary robot that the Jacobian matrix of the observation function \( H_k \) has the same values for each observation, that the Jacobian matrix of the state transition function \( F_k = I \) and that the covariance of the process noise \( Q_k = 0 \). With these assumptions, we prove the convergence of the error covariance matrix with a similar method in [5]. Here, note that the following theorems hold when the proposed filter has its unique solution and finite escape time does not occur.

We define \( P_k^{\infty} \) as the estimated error covariance matrix at time \( k \) when a mobile robot stops, and \( P_k \) as that after \( i \) times that the robot stops and an observation is made. Then, the following theorem for the determinant of the error covariance matrix is obtained.

**Theorem 1**: Suppose that Assumptions 1 and 2 hold and that a robot is stationary and a unique solution of the proposed \( H_k \) filter represented in Eqs. (20) to (25) exits. Then, Eq. (46) is a sufficient condition that the determinant of the error covariance matrix decreases monotonically when the observations are updated.

\[ W^t := H_k^T R_k^{-1} H_k - \gamma^{-2} I > 0 \quad (46) \]

**Proof**: Assuming that a robot is stationary, \( F_k = I \) and \( Q_k = 0 \). Moreover, if the robot stops, since \( H_k \) has the same value for each observation, \( W^t \) is also the same value in each observation from Eq. (46). Then, from Eq. (24), we can derive the following equation with \( W^t \).

\[ P_k^{\infty-1} = P_k^{\infty-1} + H_k^{-1} R_k^{-1} H_k^{-T} - \gamma^{-2} I = P_k^{\infty-1} + W^t \quad (47) \]

In the above equation, since \( P_k^0 \) is a positive definite matrix, if Eq. (46) is satisfied, the relationship between \( |P_k| \) and \( |\tilde{P}_k| \) is as follows.

\[ |P_k^{\infty-1}| = |(P_k^{\infty-1} + W^t)|^{-1} < |P_k^{\infty-1}|^{-1} = |\tilde{P}_k^{\infty-1}| \quad (48) \]

Similarly, since \( P_k^+ \) is a positive definite matrix, if Eq. (46) is satisfied, \( |P_k^+| \) is represented with \( |\tilde{P}_k^+| \) as follows.

\[ |P_k^+| = |(P_k^+ + W^t)|^{-1} \]

\[ = |(P_k^{\infty-1} + W^t) + W^t|^{-1} |(P_k^{\infty-1} + W^t) + W^t|^{-1} = |\tilde{P}_k^+| \quad (49) \]

We thus have proved inductively that the determinant of the error covariance matrix decreases monotonically when a robot is stationary, and Eq. (46) is satisfied.

The inverse matrix of \( P_k^{\infty} \) and each of the components of \( W^t \) are defined as follows.

\[ \tilde{P}_k^{\infty-1} = \begin{bmatrix} P_{k,11} & P_{k,12} \\ P_{k,21} & P_{k,22} \end{bmatrix} \quad (50) \]

\[ W^t = \begin{bmatrix} H_k^T R_k^{-1} H_k - \gamma^{-2} I_3 & H_k R_k^{-1} H_k^T \\ H_k^T R_k^{-1} H_k - \gamma^{-2} I_3 & H_k R_k^{-1} H_k^T \end{bmatrix} \]

\[ = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \quad (51) \]

If we assume that a robot is stationary and continues to observe landmarks, we obtain the following theorem for the convergence of the error covariance matrix.

**Theorem 2**: We assume that Assumptions 1 and 2 hold and that the robot is stationary and continues to observe the landmarks. After the stationary robot has observed the landmarks \( n > 0 \) times, the estimated error covariance matrix is as shown in Eq. (52).

\[ P_k^n = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (52) \]

where \( P_{11} \) is the error covariance matrix of the robot, \( P_{22} \) is that of the landmarks, and \( P_{12} \) and \( P_{21} \) are the cross-error covariance matrices. These are defined as follows.

\[ P_{11} = n^{-1} (Q_{11} - 

\[ P_{12} = -P_{11} Q_{12} P_{22}^{-1} \quad (54) \]

\[ P_{21} = -Q_{22}^{-1} \Omega_{21} P_{11} \quad (55) \]

\[ P_{22} = Q_{22}^{-1} \Omega_{22} + Q_{12} \Omega_{22}^{-1} + n^{-1} \Omega_{22}^0 \quad (56) \]

where \( \Omega_{ij} = n^{-1} P_{ij} + W_{ij} \).

Moreover, as \( n \to \infty \), the error covariance matrix converges to limit as \( P_{ij}^0 = \Omega_{ij} \).

**Proof**: Assuming that a robot is stationary, \( F_k = I \) and \( Q_k = 0 \). The inverse matrix of the covariance matrix after the robot has stopped and made observations \( n \) times is represented as follows.

\[ P_k^{\infty-1} = P_k^{\infty-1} + n W^t = \begin{bmatrix} n \Omega_{11} & n \Omega_{12} \\ n \Omega_{21} & n \Omega_{22} \end{bmatrix} \quad (58) \]
Then, applying the inverse matrix lemma for Eq. (58), the error covariance matrix \( \mathbf{P}_k^n \) is represented as Eq. (52), and its components are represented as Eqs. (53)–(56).

Here, if \( n \to \infty \), \( \mathbf{Q}_{ij} \) in Eq. (57) becomes \( \lim_{n \to \infty} \mathbf{Q}_{ij} = \mathbf{W}_{ij} \). In addition, \( \mathbf{P}_{11} \) in Eq. (53) converges to \( \lim_{n \to \infty} \mathbf{P}_{11} = 0 \). Therefore, since all the components of Eq. (52) converge to \( \mathbf{0} \), the estimated error covariance matrix \( \mathbf{P}_k^n \) converges to \( \lim_{n \to \infty} \mathbf{P}_k^n = 0 + 2 \mathbf{M} \). The convergence of the system has thus been proved theoretically.

5. SIMULATION

In this section, we describe a simulation to verify the usefulness of the proposed method.

5.1 Simulation Conditions

We simulated that a mobile robot moves around ten unknown landmarks and one known landmark under the parameters shown in Table 1. The velocity and turning rate of the robot are shown in Figs. 3 and 4, respectively. In this simulation, the noise were constructed from uniformly distributed random numbers within the ranges shown in Table 1.

Table 1 Simulation parameters

<table>
<thead>
<tr>
<th>Parameter [Unit]</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling Time [s]</td>
<td>SimTime</td>
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</tr>
<tr>
<td>Observation Noise Covariance Matrix</td>
<td>( \mathbf{Q} )</td>
<td>( 10^{-2} \mathbf{I} )</td>
</tr>
<tr>
<td>Initial State Error Covariance Matrix</td>
<td>( \mathbf{P}_0 )</td>
<td>( 10^{-5} \mathbf{I} )</td>
</tr>
<tr>
<td>Known Landmark Position [m]</td>
<td>( (x_0, y_0) )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Process Noise Covariance Matrix</td>
<td>( \mathbf{Q} )</td>
<td>( 10^{-5} \mathbf{I} )</td>
</tr>
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<td>Process Noise</td>
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<td>( 0.08 )</td>
</tr>
<tr>
<td>Observation Noise</td>
<td>( R_{\omega x}, R_{\omega y} )</td>
<td>( 0.001 )</td>
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<tr>
<td>Design parameter</td>
<td>( \gamma )</td>
<td>0.01</td>
</tr>
</tbody>
</table>

5.2 Simulation Results

Figures 5–8 show the estimated positions of the robot and the landmarks, the estimated error covariance matrix, and the root mean squared errors (RMSE) of the estimated robot and landmark positions, respectively.

First, in Fig. 5, the lines and markers represent the robot paths and the landmark positions, respectively, and the star shows the position of the known landmark. From this figure, the estimations with our proposed method were carried out with a good estimation accuracy until the end of the simulation. In Fig. 6, we see that the covariance for the \( H_\infty \) filter with a known landmark (HF with KL) is suppressed compared with that for the conventional \( H_\infty \) filter (HF). Thus Lemma 2 is confirmed. The estimation accuracy was evaluated in detail by comparing the estimation errors. Figure 7 and 8 show the RMSE of the positions of the robot and landmarks, respectively. From both these two figures, we see that the errors from our proposed method (HF with KL) were less than those from the EKF and the HF. Therefore, we can verify that the estimation accuracy was improved by the use of the \( H_\infty \) filter with the observation of a known landmark.

6. EXPERIMENT

Next, we validated the proposed method with an experiment using the simulation program and the data obtained from an actual robot, the Amigobot.

6.1 Experimental setup and conditions

Figure 9 shows the setup and the overview of the experiment. Experimental verification was carried out under the parameters shown in Table 2. We gave the input signals shown in Fig. 10 to the Amigobot through a wireless LAN and allowed the robot to move around the landmark in a circle with a radius of one meter. In addition, we set the \( a \) priori known landmark at \( [x_0, y_0] = [400, 200] \). The Amigobot obtained the relative distances to the landmarks with sonar sensors. We used Matlab to calculate the estimated robot trajectory and the positions of the landmarks with each of the filters and to compare them with their true values obtained by using a camera set above the equipment.
6.2 Experimental Results

The results are shown in Figs. 11–13, which are the estimated positions of the robot and the landmark, the estimated error covariance matrices, and the root mean squared errors (RMSE) of the position of the landmark. Since the Amigobot and the raised camera to obtain true values worked independently of each other, we could not synchronize their sampling times. Thus the RMSE of the robot position in each step was omitted.

First, in Fig. 11, the estimations of both the robot path and a landmark position using the $H_{\text{ef}}$ filter with the observation of a known landmark (HF with KL) are closer to the true value than the estimations of the other filters. Next, comparing the error covariance matrices in Fig. 12, we see that the error covariance when using the conventional $H_{\text{ef}}$ filter diverged to negative immediately after the experiment began, and finite escape time occurred. This was avoided by using our proposed method. Moreover, this figure also shows that the error covariance matrix with the proposed method is suppressed and that it continued to converge until the end of the experiment. In Fig. 13, though the error when using the conventional $H_{\text{ef}}$ filter became large, the error with our proposed method did not become large, and it was less than that with the EKF. We thus confirm that the estimation accuracy is improved by using our proposed method.

![Fig. 11 Estimation result](image)

![Fig. 12 Error covariance](image)

![Fig. 13 Errors of landmarks](image)

7. CONCLUSION

In this paper, we described the SLAM problem via the $H_{\text{ef}}$ filter with a known landmark. In addition to observations for unknown landmarks, this proposed method updates its estimated values with an observation for a landmark that location is known a priori. Including the observation for a known landmark, the system satisfies observability and its estimation accuracy is improved. In this paper, we proved this improvement theoretically by showing that the determinant of the error covariance matrix becomes small with the observation for a known landmark. We also proved the convergence of the estimated error covariance matrix for the proposed algorithm for the case of updates only when the robot is stationary. Finally, simulation and experimental results shows that the reduction of the estimated error covariance matrix and its convergence. We also confirmed that our proposed method improved the estimation accuracy compared to other filters in these results.

REFERENCES


